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JEL CLASSIFICATION: D81, D82, D83

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1 Introduction

Imagine a sender (he) who can provide some information to influence the decision making of a rational Bayesian receiver (she) who has a private source of information. The sender only has limited knowledge about what the receiver privately knows and wants to design a persuasion rule that is robust to this uncertainty. Can the sender gain from persuasion? What is the optimal way to persuade?

The above question is relevant in many economic settings. For example, when a bond rating agency chooses what information to disclose to investors about bond issuers, he knows that investors may also have access to private information from other channels, such as newspapers and the Internet. The agency's knowledge about the investors' private information is limited in the sense that the agency knows the possible channels from which the private information is generated, but does not know from which channel a particular investor obtains her information. In another example, when a school chooses what information to disclose on transcripts to prospective employers about the ability of its students, it knows that employers may also obtain private information from the studetns' extracurricular activities. The school knows the set of all possible extracurricular activities, but does not know which particular extracurricular activity is observed by an employer.

Our model is built on Kamenica and Gentzkow (2011). There is a sender who designs a disclosure rule to convey information about the state of the world, and a receiver who takes an action that affects her and the sender's payoffs. The sender and receiver initially hold common prior beliefs about the state of the world. In addition, we assume that the receiver also receives private information from her private source, about which the sender has only limited knowledge. We model the receiver's private information source as an information structure and formulate the sender's limited knowledge about the receiver's private information source as a *belief-based* collection of information structures, which is the set of all possible information structures satisfying that the receiver's private belief, updated from the common prior and her private signal drawn from any information structure, is contained in a certain convex range $\widehat{\Delta}$. The sender neither observes the receiver's private signal nor is aware of from which information structure it is generated, but only knows the range of the receiver's private beliefs $\widehat{\Delta}$, and thinks that every information structure in this belief-based collection is possibly the receiver's private information structure.

We investigate how an ambiguity-averse sender with maxmin expected utility optimally designs his robust information disclosure rule. The maxmin expected utility criterion,[1] which is perhaps the most commonly adopted model in previous studies of models involving ambiguity and robustness concerns, simply requires that the sender evaluates each information disclosure rule by the worst possible expected payoff generated by the receiver's private information structure from the belief-based collection. The optimal persuasion rule is chosen so that it maximizes this worst expected payoff.

Our first step is to reformulate the receiver's private information structures and the sender's information design. For a fixed prior belief, Kamenica and Gentzkow (2011) prove the equivalence between information structures and posterior belief distributions consistent with this prior belief. We follow this approach and reformulate the sender's uncertainty about the receiver's private information structure as the sender's uncertainty about the distribution of the receiver's private beliefs. Any belief-based collection of information structures is equivalent to the collection of posterior belief distributions whose mean is the common prior and whose support is contained in some $\widehat{\Delta}$. As for the sender's information design, unlike Kamenica and Gentzkow (2011), we find it more convenient to work directly with information structures than posterior belief distributions. This is because in our model the sender cares about not only the receiver's posterior belief distribution updated from the common prior, but also the posterior belief distributions updated from various private beliefs. We use *Blackwell's standard information structures* (Blackwell (1951, 1953)) to narrow down the sender's choice space. This allows us to easily write out, for each designed information structure, the sender's *contingent payoff function* — his expected payoff from this information design as a function of the receiver's private belief.

To characterize the sender's optimal information design, we derive a novel convexification result, which shows that the sender's worst case expected payoff from an information design simply equals the value of the largest convex function below his contingent payoff function over $\widehat{\Delta}$ evaluated at the common prior. Based on this intermediate result, we study the sender's optimal value of persuasion in two cases according to the sender's degree of ambiguity. One is *full ambiguity*, where the sender thinks that every information structure that the receiver may have is possible; the other is *local ambiguity*, where the sender thinks that the receiver's private beliefs are all close to the common prior.

In the full ambiguity case, similar to the findings in many papers with robustness concerns (e.g., Garrett (2014) and Carroll (2015)), the sender's optimal value can be achieved by a very simple rule: fully revealing the states to the receiver is always optimal for the sender. This is because the sender's worst case expected payoff from

[1] See Gilboa and Schmeidler (1989) for an axiomatic representation of this preference.

any information design cannot be higher than his expected payoff when the receiver fully observes the states, since the sender cannot change the receiver's action in this case. By fully revealing the states to the receiver, the sender can guarantee himself this upper bound regardless of the receiver's private information. Thus, full information disclosure is optimal. Based on this, our next result then shows that the sender can gain from persuasion no matter what the interior initial prior is if and only if there exists at least one private belief such that full disclosure makes him strictly better off than letting the receiver choose her default action. This is in contrast to the model of Kamenica and Gentzkow (2011), as there is no gain from persuasion when doing so makes the sender no better off than letting the receiver choose her default action. In our framework with robustness concerns, the sender can still gain from persuasion because doing so can avoid the possible unfavorable private belief induced by the receiver's private signal.

In the local ambiguity case, we obtain an intuitive continuity result by showing that for generic payoffs, as the sender's uncertainty about the receiver's private information vanishes, the sender's optimal value converges to his optimal value when the receiver does not have private information.[2] However, this convergence result is not because the sender's optimal persuasion rule is robust to small uncertainty. The optimal persuasion rule when the receiver does not have private information (and hence the receiver's private belief is the same as the common prior) is generically not robust. This is because some signals of that optimal persuasion rule usually make the receiver indifferent between several actions when the receiver holds the common prior. Even if the receiver's private belief is only a small perturbation of the common prior, the receiver's behavior will change dramatically, possibly resulting in a much lower payoff to the sender. Nevertheless, we construct information disclosure rules that can guarantee the sender values arbitrarily close to his optimal value when the receiver does not have private information, provided that the uncertainty is small. Consequently, the sender can gain from persuasion in the face of local ambiguity if he can do so when there is no ambiguity.

Finally, we provide a novel method to fully characterize the optimal persuasion rule for various cases of sender's ambiguity in the prosecutor-judge leading example of Kamenica and Gentzkow (2011). In this example, there are two states, $i = 1$ and $i = 2$, and two actions, $a = 1$ and $a = 2$. The sender always prefers $a = 2$, while the receiver takes $a = 2$ only if her belief about state $i = 2$ is greater than or equal to $\frac{1}{2}$. Because there are only two states, we can identify the sender's

[2]For non-generic payoffs, this result does not hold. See Appendix B for a counterexample.

ambiguity by an interval $[\alpha, \beta]$ containing the prior, which represents the range of the receiver's private beliefs on the state $i = 2$. Clearly, if $\alpha \geq \frac{1}{2}$, it is always optimal for the sender not to reveal any information because the receiver will take the sender's preferred action by default regardless of her private signal. However, when $\alpha < \frac{1}{2}$, the characterization of the sender's optimal persuasion rule becomes much more involved. The key insight from our characterization is that the sender's optimal value from persuasion is always achieved by a *linear-contingent-payoff information structure*, i.e., an information structure under which the sender's contingent payoff function over $[\alpha, \beta]$ takes the form $\max\{0, \ell\}$ for some linear function ℓ. By choosing a linear-contingent-payoff information structure, the sender fully insures himself against any uncertainty about the receiver's private information structure, because the sender's worst case expected payoff from such contingent payoff function is simply the function itself as implied by our convexification result. We then identify a cutoff belief between α and β. If the common prior is lower than this cutoff, the sender's optimal persuasion rule coincides with his optimal rule in Kamenica and Gentzkow (2011) with prior α. If the common prior is above this cutoff, his optimal rule is a different linear-contingent-payoff information structure that would give up persuasion when the receiver's private belief is low in exchange for a higher chance of persuasion when the receiver's private belief is high.

1.1 Related Literature

Our Bayesian persuasion model is a variation of Kamenica and Gentzkow (2011), with the new ingredient that the receiver is privately informed and the sender only has limited knowledge about the receiver's private information source. We study how a sender optimally reveals information that is robust to the receiver's private information. Bayesian persuasion of a privately informed receiver has been studied in Rayo and Segal (2010), Kamenica and Gentzkow (2011), Kolotilin et al. (2016), Kolotilin (forthcoming) and Guo and Shmaya (2017).[3] These papers all assume that the distribution of the receiver's private information is common knowledge as in the usual

[3]Many other papers have studied Bayesian persuasion problems in various contexts. For example, Gill and Sgroi (2012), Perez-Richet (2014) and Hedlund (2017) study Bayesian persuasion with a privately informed sender; Gentzkow and Kamenica (2014) consider persuasion with costly signals; Alonso and Câmara (2016) explore a Bayesian persuasion problem without a common prior; Gentzkow and Kamenica (2017a) and Li and Norman (2017) analyze competition in Bayesian persuasion; and Gentzkow and Kamenica (2017b) investigate endogenous information acquisition of information in a Bayesian persuasion environment; to name a few.

mechanism design literature, but we consider the environment in which the sender thinks many distributions are possible. While Rayo and Segal (2010), Kolotilin et al. (2016), Kolotilin (forthcoming) and Guo and Shmaya (2017) model the receiver's private information as her private preference and the last three also consider private persuasion (as called by Kolotilin et al. (2016)), our model is closest to Section VI.A of Kamenica and Gentzkow (2011). We model the receiver's private information as her private belief, and focus on public persuasion in which the sender designs a single information disclosure rule for all receiver types. Because the sender in our model is uncertain about the receiver's private information source, he cannot simply form an expectation of the receiver's private beliefs by "integrating over the receiver's private signal," as suggested in Section VI.A of Kamenica and Gentzkow (2011). Consequently, the standard *concavification* approach of Kamenica and Gentzkow (2011) does not apply in our model. Instead, we rely on a new *convexification* approach to derive our characterizations (see Lemma 4).

Our paper is also related to the growing literature on robust mechanism design under ambiguity aversion. The literature has studied various contexts, such as auction design, bilateral trade, monopoly pricing, and moral hazard.[4] To our best knowledge, our paper is the first to investigate robust Bayesian persuasion of a privately informed receiver.[5] Moreover, in the previous literature, the principal is completely uncertain about the distributions or only knows some moments of the distributions (e.g., Carrasco et al. (2017)). In our setup, the principal (the sender) not only knows the mean of the distributions, but also may have further knowledge about the support of the distributions. We believe that the general method developed in this paper can also be applied to study the robust Bayesian persuasion of a privately informed receiver

[4]An incomplete list of these studies includes Bergemann and Schlag (2008), Bergemann and Schlag (2011) and Carrasco et al. (2017), who study monopoly pricing when the monopolist only has limited knowledge about the distribution of the buyer's valuation; Garrett (2014) analyzes a model of cost-based procurement where the seller is uncertain about the agent's effort cost function; Carroll (2015) considers a principal-agent model in which the principal is uncertain about what the agent can and cannot do; Bose et al. (2006), Bodoh-Creed (2012) and Bose and Renou (2014) investigate auction design problems in which each bidder is uncertain about other bidders' valuation distributions; and Wolitzky (2016) studies efficiency in a bilateral trade model in which the seller and buyer only know the mean of each other's valuations. More recently, Du (2017) and Bergemann et al. (2016) study robust common value auction design in which the seller is uncertain about the bidders' information structures.

[5]Following Bose and Renou (2014), Li and Li (2017) consider ambiguous persuasion where the sender can send a signal with multiple likelihood distributions. But this possibility is not allowed in our model.

in other frameworks, such as Rayo and Segal (2010) and Kolotilin et al. (2016), and other robust mechanism design issues in similar contexts.

2 Model

2.1 Basic setup and notation

Let $\Omega = \{1, 2, \ldots, N\}$ be the set of states of the world. There is a sender and a receiver. The sender designs information and the receiver takes an action a from a finite set A. The sender's ex post payoff is $v : A \times \Omega \to \mathbb{R}$ and the receiver's is $u : A \times \Omega \to \mathbb{R}$. At the beginning of the game, the sender and receiver share a common prior $\pi \in \Delta(\Omega) = \Delta^{N-1}$.[6]

The receiver can receive private information from her private information source. We model sender's information design and the receiver's private information source by *information structures*. An information structure (equivalently, a statistical experiment) $I = (S, \mu_1, \cdots, \mu_N)$ consists of a Borel measurable set S of signals and conditional distributions of signals: $\mu_i \in \Delta(S)$ for each $i \in \Omega$. For each $I = (S, \mu_1, \ldots, \mu_N)$, let $\mu_0 \equiv (\sum_i \mu_i)/N$ be the "average distribution" of signals. Given an initial belief $p \in \Delta^{N-1}$ over the states of the world and an information structure $I = (S, \mu_1, \ldots, \mu_N)$, each signal realization $s \in S$ leads to an updated belief via Bayes' rule:

$$q^I(p, s) \equiv \Big(\frac{p_1 f_1(s)}{\sum_i p_i f_i(s)}, \ldots, \frac{p_N f_N(s)}{\sum_i p_i f_i(s)}\Big) \in \Delta^{N-1}, \tag{1}$$

where $f_i : S \to \mathbb{R}$ is the Radon-Nikodym derivative (or density) of μ_i with respect to μ_0. Moreover, for each information structure $I = (S, \mu_1, \ldots, \mu_N)$ and an initial belief $p \in \Delta^{N-1}$, we write $\mu^p \equiv \sum_i p_i \mu_i$ as the *unconditional distribution of signals* and $\widehat{\mu}^p \equiv \mu^p \circ q^I(p, \cdot)^{-1}$ as the *unconditional distribution of updated beliefs*.[7] Finally, let $\mathcal{I}$ denote the set of all information structures.

2.2 Modeling the sender's ambiguity

Suppose the receiver's private information structure is $I_r = (S_r, \nu_1, \ldots, \nu_N)$. After observing a realization $s_r \in S_r$, the receiver's updated *private belief* becomes $q^{I_r}(\pi, s_r)$.

[6] If X is a Borel measurable set X, $\Delta(X)$ denotes the set of all probability measures over X.

[7] Given two measurable sets X and Y, a measurable function $g : X \to Y$ and a measurable ν over X, $\mu \circ f^{-1}$ denotes the push-forward measure of μ, i.e. $\mu \circ f^{-1}(Y') \equiv \mu(\{x \in X | f(x) \in Y'\})$ for all measurable $Y' \subseteq Y$.

If the sender knows I_r, he can form an expectation about the distribution of the receiver's private beliefs even if he cannot observe the realized signal, as discussed in Kamenica and Gentzkow (2011). However, we assume that when designing information, the sender neither observes the receiver's signal nor is aware of her private information structure. The only knowledge that the sender has is that the receiver's private information structure belongs to a certain collection of information structures $\widehat{\mathcal{I}} \subseteq \mathcal{I}$. Ambiguity arises as the sender does not even know the receiver's private information structure, and hence $\widehat{\mathcal{I}}$ represents the degree of ambiguity faced by the sender. We assume that the sender's knowledge is correct in the sense that the receiver's true private information structure is indeed contained in $\widehat{\mathcal{I}}$. Thus, this rules out the situations where the sender completely misspecifies the receiver's private information structure.[8]

There are potentially many different ways to model the sender's ambiguity, $\widehat{\mathcal{I}}$. In this paper, we will focus on one particular approach, which directly links $\widehat{\mathcal{I}}$ to the set of the receiver's private beliefs. We say a collection $\widehat{\mathcal{I}} \subseteq \mathcal{I}$ of information structures is *belief based* if there exists a nonempty, convex and compact subset $\widehat{\Delta} \subseteq \Delta^{N-1}$ such that $\pi \in \widehat{\Delta}$ and[9]

$$\widehat{\mathcal{I}} = \{I \in \mathcal{I} \big| \mathrm{supp}(\widehat{\mu}^{\pi}) \subseteq \widehat{\Delta}\}.$$

In this case, we explicitly write $\widehat{\mathcal{I}}$ as $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$. In words, $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$ contains all the information structures whose induced private beliefs are all contained in $\widehat{\Delta}$, given the common prior π. Thus, a belief-based collection of information structures captures the idea that the sender does not know the receiver's private information structure but believes that after observing her private signal, the receiver forms a private belief that is in a certain range no matter what private information structure the receiver actually has and what signal realization she observes.

The following two examples are two extreme cases of belief-based collections of information structures.

Example 1. $\widehat{\Delta} = \{\pi\}$. This corresponds to the situation where the sender knows that the receiver's private belief after observing her private signal is always the common prior π regardless of the signal realization. Thus, $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$ contains only null information structures, i.e., information structures whose signals are completely un-

[8] See, for example, Esponda and Pouzo (2016a,b) for recent discussions of solution concepts under model misspecification.

[9] The requirement that $\pi \in \widehat{\Delta}$ is innocuous because otherwise $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$ is an empty set. See the discussion before Lemma 1.

informative. In this case, there is in fact no ambiguity and our model degenerates to that in Kamenica and Gentzkow (2011).

Example 2. $\widehat{\Delta} = \Delta^{N-1}$. This corresponds to the situation where the sender has no knowledge at all about the receiver's private information structure and thinks that the receiver's private signal can lead to any potential private belief. Thus, $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi) = \mathcal{I}$. In this case, there is *full ambiguity* and the sender thinks that every information structure that the receiver may have is possible.

The following is an example in which the sender has partial knowledge about the information structure from which the receiver's signal is generated.

Example 3. $\widehat{\Delta} = \{q \in \Delta^{N-1} | q_i \geq \alpha_i \text{ for } i = 1, \dots, N\}$ for some $\alpha_1, \dots, \alpha_N > 0$ and $\sum_i \alpha_i < 1$. That is, although the sender is uncertain about what private information structure the receiver has, he knows that the receiver's private signal always leads to an interior private belief that is bounded away from the boundary of Δ^{N-1}. In other words, the sender is sure that the receiver cannot receive very precise information about the states. In particular, when $N = 2$, this corresponds to the situation where the sender believes that the receiver's private information has bounded likelihood ratios, i.e.,

$$\widehat{I}(\widehat{\Delta}, \pi) = \left\{ I \in \mathcal{I} \,\middle|\, \frac{\pi_1 \alpha_2}{\pi_2(1-\alpha_2)} \leq \frac{f_2(s)}{f_1(s)} \leq \frac{(1-\alpha_1)\pi_1}{\pi_2 \alpha_1},\ \forall s \in S \right\}.$$

2.3 Sender's information design problem

Aside from the receiver's private information, the sender can design an information structure to supply supplemental information to the receiver. We assume throughout this paper that the receiver's private information and the sender's information are conditionally (on states) independent (as in Kamenica and Gentzkow (2011) and Bergemann et al. (forthcoming)).

The timing of the game is as follows. The sender first chooses an information structure and commits himself to revealing whatever signal he observes; the receiver then observes her private signal and the sender's signal, and makes an action choice. For any $q \in \Delta^{N-1}$, let $a[q] = \arg\max_{a \in A} \sum_i q_i u(a, i)$ be the receiver's optimal action choice if her posterior belief after observing these two signals is q. Following Kamenica and Gentzkow (2011), we consider the sender-preferred subgame perfect equilibrium: if the receiver is indifferent between some actions at a given belief, she takes an action that maximizes the sender's expected payoff under this belief.

Suppose the receiver's private information structure is $I_r = (S_r, \nu_1, \ldots, \nu_N)$ and the sender's information structure is $I_s = (S_s, \mu_1, \ldots, \mu_N)$. Because the two information structures are independent, the receiver's *posterior belief* after observing a signal s_r from I_r and a signal s_s from I_s becomes $q^{I_s}(q^{I_r}(\pi, s_r), s_s) \in \Delta^{N-1}$, and thus she optimally chooses $a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big] \in A$. Therefore, if I_r was publicly known, the sender's ex-ante expected payoff from designing information structure I_s would be

$$\begin{aligned}
&\sum_i \pi_i \int_{S_r} \Big(\int_{S_s} v\big(a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu_i(\mathrm{d}s_r) \\
&= \int_{S_r} \sum_i \pi_i \frac{\mathrm{d}\nu_i}{\mathrm{d}\nu_0} \times \Big(\int_{S_s} v\big(a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu_0(\mathrm{d}s_r) \\
&= \int_{S_r} (\sum_j \pi_j \frac{\mathrm{d}\nu_i}{\mathrm{d}\nu_0}) \times \Big(\sum_i q_i^{I_r}(\pi, s_r) \int_{S_s} v\big(a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu_0(\mathrm{d}s_r) \\
&= \int_{S_r} \sum_i q_i^{I_r}(\pi, s_r) \Big(\int_{S_s} v\big(a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu^\pi(\mathrm{d}s_r),
\end{aligned} \tag{2}$$

where, $\nu_0 \equiv (\sum_i \nu_i)/N$ is the average distribution of the receiver's private signals, and $\nu^\pi \equiv \sum_i \pi_i \nu_i$ is the unconditional distribution of the receiver's private signals given prior π. The second equality of the above expression comes from (1).

However, the receiver's information structure is private and the sender is uncertain about it. The sender only knows that the receiver's private information structure is one of those in $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$. Following the standard *maxmin expected utility function* assumption in the ambiguity aversion literature (e.g., Gilboa and Schmeidler (1989), Garrett (2014), Carroll (2015)), we assume that the sender evaluates an information structure I_s by its worst case expected payoff. When the sender designs I_s, it is the worst case expected payoff that he seeks to maximize. Formally, for each $I_s \in \mathcal{I}$, let

$$V^{I_s}(\widehat{\Delta}, \pi) \equiv \frac{1}{N} \inf_{I_r \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)} \int_{S_r} \Big(\sum_i q_i^{I_r}(\pi, s_r) \int_{S_s} v\big(a\big[q^{I_s}(q^{I_r}(\pi, s_r), s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu^\pi(\mathrm{d}s_r) \tag{3}$$

be the sender's (normalized) worst case expected payoff if he designs information structure I_s, where $1/N$ is a normalization. The sender's problem can then be succinctly written as

$$V(\widehat{\Delta}, \pi) \equiv \max_{I_s \in \mathcal{I}} V^{I_s}(\widehat{\Delta}, \pi). \tag{4}$$

3 Simplifying the sender's problem

The sender's problem in (4) is in general not easy to work with because the space $\mathcal{I}$ (and hence $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$) is very large and abstract. In this section, we try to simplify it by reformulating the receiver's private information structures and the sender's information design.

In considering the sender's information design problem in their model, Kamenica and Gentzkow (2011) prove that the sender's choice of information structure is equivalent to choosing a distribution over posterior beliefs whose mean is the common prior, because every information structure induces a posterior belief distribution with the mean being the common prior and vice versa. Here, we apply this result to the reformulation of the *receiver's private information structures.* We show in Section 3.1 that the sender's uncertainty about the receiver's private information structure can be equivalently modeled as the sender's uncertainty about the *distribution of the receiver's private beliefs.* The sender's information design is more subtle in our model, because the receiver may have private beliefs that are different from the common prior and consequently the sender cares about posterior belief distributions induced by not only the common prior, but also various private beliefs. As a result, we find it easier to work directly with information structures, and we use *Blackwell's standard information structure* (Blackwell (1951, 1953)) to narrow down the choice space of the sender.[10]

3.1 Receiver's private belief distributions

For any receiver's private information structure $I_r = (S_r, \nu_1, \ldots, \nu_N) \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$, recall that $\widehat{\nu}^\pi$ is the unconditional distribution of the receiver's private beliefs given prior π. Using this notation, we can rewrite the sender's expected payoff from information structures I_r and I_s in (2) as

$$\int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{S_s} v\big(a\big[q^{I_s}(p, s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \widehat{\nu}^\pi(\mathrm{d}p). \tag{5}$$

From (5), it is obvious that each receiver's private information structure $I_r \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$ affects the sender's expected payoff only through $\widehat{\nu}^\pi$. Since $I_r \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$, we know $\mathrm{supp}(\widehat{\nu}^\pi) \subseteq \widehat{\Delta}$ by definition. Moreover, because $\widehat{\nu}^\pi$ is the distribution of posteriors updated from prior π via Bayes' rule, it is a standard result that the mean of $\widehat{\nu}^\pi$ is simply π, i.e., $\int_{\Delta^{N-1}} p \widehat{\nu}^\pi(\mathrm{d}p) = \pi$. The following lemma, which is a simple extension of Proposition 1 in Kamenica and Gentzkow (2011), states that the converse is also

[10] Another equivalent approach is to model the sender's choice of information structure as choosing a posterior belief distribution given the common prior and, for the same reason explained above, directly work with information structures for the receiver's private information structures. But we find that this formulation is not very helpful.

true: the support and mean restrictions are the only restrictions on the set of all distributions of posterior beliefs that can be induced by an information structure in $\widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$.

Lemma 1. *Fix $\widehat{\Delta}$ and $\pi \in \widehat{\Delta}$. For any probability distribution $\nu \in \Delta(\Delta^{N-1})$ with* $\mathrm{supp}(\nu) \subseteq \widehat{\Delta}$ *and $\int_{\Delta^{N-1}} p\nu(\mathrm{d}p) = \pi$, there exists an information structure $I \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$ such that $\widehat{\nu}^{\pi} = \nu$.*

For any $\widehat{\Delta}$ and $\pi \in \widehat{\Delta}$, let $\mathcal{G}(\widehat{\Delta}, \pi)$ be the set of all probability distributions over Δ^{N-1} whose support is contained in $\widehat{\Delta}$ and whose mean is π. Formally

$$\mathcal{G}(\widehat{\Delta}, \pi) \equiv \{\nu \in \Delta(\Delta^{N-1}) | \mathrm{supp}(\nu) \subseteq \widehat{\Delta} \text{ and } \int_{\Delta^{N-1}} p\nu(\mathrm{d}p) = \pi\}.$$

Lemma 1 then states that $\mathcal{G}(\widehat{\Delta}, \pi) = \{\widehat{\nu}^{\pi} | I \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)\}$. Therefore, plugging expression (5) into (3), we can rewrite the sender's worst case expected payoff from information structure I_s as

$$\begin{aligned}
V^{I_s}(\widehat{\Delta}, \pi) &= \frac{1}{N} \inf_{I_r \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{S_s} v\big(a\big[q^{I_s}(p, s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \widehat{\nu}^{\pi}(\mathrm{d}p) \\
&= \frac{1}{N} \inf_{\nu \in \{\widehat{\nu}^{\pi} | I_r \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)\}} \int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{S_s} v\big(a\big[q^{I_s}(p, s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu(\mathrm{d}p) \\
&= \frac{1}{N} \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{S_s} v\big(a\big[q^{I_s}(p, s_s)\big], i\big) \mu_i(\mathrm{d}s_s) \Big) \nu(\mathrm{d}p). \qquad (6)
\end{aligned}$$

3.2 Sender's standard information structure

The sender's information design problem is still not easy to work with because the set of all information structures is very rich. We now simplify the sender's information design problem by narrowing down the sender's choice space. We say two information structures $I = (S, \mu_1, \ldots, \mu_N)$ and $I' = (S', \mu'_1, \ldots, \mu'_N)$ are *equivalent* if these two induce the same *conditional distributions of posterior beliefs for every prior*, i.e., for all $p \in \Delta^{N-1}$, $\mu_i \circ q^I(p, \cdot)^{-1} = \mu'_i \circ q^{I'}(p, \cdot)^{-1}$ for all $i = 1, \ldots, N$.

Clearly, if two information structures are equivalent, they always induce the same unconditional distribution of posterior beliefs for any prior. Thus, from the sender's point of view, if two information structures are equivalent, then they always lead to identical receiver's posterior belief distributions no matter what the receiver's private belief is. This in turn implies that although the sender does not know the distribution of the receiver's private belief, two equivalent information structures will always yield the same expected payoff to the sender. This simple observation is summarized in the following lemma.

Lemma 2. *If I_s and I_s' are equivalent, then $V^{I_s}(\widehat{\Delta},\pi) = V^{I_s'}(\widehat{\Delta},\pi)$ for all $\widehat{\Delta}$ and π.*

In view of Lemma 2, we only need to focus on one representative for each class of equivalent information structures. Conceptually, the choice of the representative can be arbitrary. But we find Blackwell's *standard information structures* to be particularly easy to work with as representatives.

Definition 1 (Blackwell (1951))**.** An information structure $I = (S, \mu_1, \ldots, \mu_N)$ is a *standard information structure* if $S = \Delta^{N-1}$, and for all $i = 1, \ldots, N$,

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}\mu_0}(s) = Ns_i,\ \mu_0 - a.s. \tag{7}$$

Standard information structures all use Δ^{N-1}, the set of all beliefs, as their signal space. Moreover, if $I = (S, \mu_1, \ldots, \mu_N)$ is a standard information structure, then condition (7) implies that the posterior belief updated via Bayes' rule from prior $p \in \Delta^{N-1}$ and signal $s \in S$ is

$$q^I(p,s) \equiv \Big(\frac{p_1 s_1}{\sum_i p_i s_i}, \ldots, \frac{p_N s_N}{\sum_i p_i s_i}\Big) \in \Delta^{N-1}. \tag{8}$$

Notice that the posterior belief for a given prior $p \in \Delta^{N-1}$ and a signal $s \in \Delta^{N-1}$ is the same for all standard information structures because the right hand side of (8) is independent of I. For this reason, throughout the paper, we suppress the superscript I and simply write $q(r,s)$ as the posterior belief updated from a standard information structure.

The following result, due to Blackwell (1951) (see also Blackwell (1953)), states that the collection of all standard information structures is rich enough to "represent" all information structures.[11]

Lemma 3 (Blackwell (1951, 1953))**.** *Every information structure $I \in \mathcal{I}$ is equivalent to some standard information structure.*[12]

Consider a standard information structure I. It is immediate from condition (7) that $1 = \mu_i(\Delta^{N-1}) = \int_{\Delta^{N-1}} Ns_i\mu_0(\mathrm{d}s)$ or equivalently $\int_{\Delta^{N-1}} s_i\mu_0(\mathrm{d}s) = 1/N$ for all $i = 1, \ldots, N$. That is, the "average signal" of a standard information structure

[11] Smith and Sørensen (2000) also used this fact to model the agents' private signals in their model of observational learning. See their Appendix A.

[12] The notion of equivalent information structures in Blackwell (1951) is different from the one we give in the current paper. But it is easy to see that these two notions are equivalent. In the Appendix, we briefly discuss how Lemma 3 can be verified directly.

given equal prior is $(1/N, \dots, 1/N)$.[13] On the other hand, every probability measure $\mu \in \Delta(\Delta^{N-1})$ with mean $(1/N, \dots, 1/N)$ induces a standard information structure. To see this, define μ_i according to the Radon-Nikodym derivative $\frac{\mathrm{d}\mu_i}{\mathrm{d}\mu}(s) = Ns_i$ for $s \in \Delta^{N-1}$. Then it is straightforward to verify that $(\Delta^{N-1}, \mu_1, \dots, \mu_N)$ is indeed a standard information structure and $\mu = (\sum_i \mu_i)/N$. This suggests that the set of all standard information structures can be characterized by

$$\mathcal{F} \equiv \Big\{\mu \in \Delta(\Delta^{N-1}) \Big| \int_{\Delta^{N-1}} s_i \mu(\mathrm{d}s) = 1/N \text{ for } i = 1, \dots, N\Big\}.$$

For this reason, with a slight abuse of notation, we also call a measure $\mu \in \mathcal{F}$ a standard information structure.

Consider a standard information structure $\mu \in \mathcal{F}$. The sender's payoff in (6) from μ can then be written as

$$\begin{aligned} V^{\mu}(\widehat{\Delta}, \pi) &= \frac{1}{N} \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{\Delta^{N-1}} v\big(a\big[q(p,s)\big], i\big) \mu_i(\mathrm{d}s)\Big) \nu(\mathrm{d}p) \\ &= \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \Big(\int_{\Delta^{N-1}} \big[\sum_i p_i s_i v\big(a\big[q(p,s)\big], i\big)\big] \mu(\mathrm{d}s)\Big) \nu(\mathrm{d}p), \end{aligned}$$

where the second equality comes from condition (7). Moreover, according to Lemmas 2 and 3, it is without loss of generality to restrict attention to standard information structures in the sender's problem. Therefore, the sender's information design problem in (4) can be reformulated as a problem of choosing a probability measure over Δ^{N-1} with the constraint that its mean is $(1/N, \dots, 1/N)$:

$$V(\widehat{\Delta}, \pi) = \max_{\mu \in \mathcal{F}} \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \Big(\int_{\Delta^{N-1}} \sum_i p_i s_i v\big(a\big[q(p,s)\big], i\big) \mu(\mathrm{d}s)\Big) \nu(\mathrm{d}p). \tag{9}$$

3.3 Convexification

For each standard information structure $\mu \in \mathcal{F}$, define the sender's *contingent payoff function* $\phi^{\mu} : \Delta^{N-1} \to \mathbb{R}$ as

$$\phi^{\mu}(p) \equiv \int_{\Delta^{N-1}} \sum_i p_i s_i v\big(a\big[q(p,s)\big], i\big) \mu(\mathrm{d}s), \ \forall p \in \Delta^{N-1}. \tag{10}$$

[13] This can also be understood from the perspective of the distribution of posterior beliefs with equal prior. From condition (7), we know $q(p,s) = s$ if $p = (1/N, \dots, 1/N)$. Therefore, the fact that $\int_{\Delta^{N-1}} s_i \mu_0(\mathrm{d}s) = 1/N$ for all $i = 1, \dots, N$ simply states that the mean of the posterior distribution is equal to the prior. From this point of view, choosing a standard information structure is simply choosing a posterior belief distribution. This is in the same spirit as Kamenica and Gentzkow (2011) with the difference that we directly focus on posterior belief distributions consistent with uniform prior instead of π.

Observe that $\phi^\mu(p)$ is the sender's expected payoff from information structure μ contingent on the receiver's private belief being p. For each $\widehat{\Delta}$, let $\phi^\mu|_{\widehat{\Delta}} : \widehat{\Delta} \to \mathbb{R}$ be the function ϕ^μ restricted to the domain $\widehat{\Delta}$. Define the *convexification of* ϕ^μ *on the domain* $\widehat{\Delta}$, denoted $\mathrm{co}_{\widehat{\Delta}}\phi^\mu : \widehat{\Delta} \to \mathbb{R}$, as the *largest convex function on* $\widehat{\Delta}$ *below* $\phi^\mu|_{\widehat{\Delta}}$. Formally, $\mathrm{co}_{\widehat{\Delta}}\phi^\mu$ is the pointwise supremum of all convex functions on $\widehat{\Delta}$ that are below $\phi^\mu|_{\widehat{\Delta}}$, i.e.,

$$\mathrm{co}_{\widehat{\Delta}}\phi^\mu(p) \equiv \sup_{\substack{\text{convex } f:\widehat{\Delta}\to\mathbb{R} \\ f \le \phi^\mu|_{\widehat{\Delta}}}} f(p), \ \forall p \in \widehat{\Delta}.$$

Figure 1 provides an illustration of convexification for a one-dimensional function. It is worth emphasizing that the value of the convexification in general depends on its domain $\widehat{\Delta}$. Specifically, the convexification of ϕ^μ on a domain $\widehat{\Delta}$ is in general different from the convexification of ϕ^μ on Δ^{N-1} restricted to $\widehat{\Delta}$. The dotted blue line in Figure 1 is the convexification over the whole domain $[0, 1]$, while the solid blue line is the convexification over the interval $[\alpha, \beta]$. Obviously, these two are quite different.

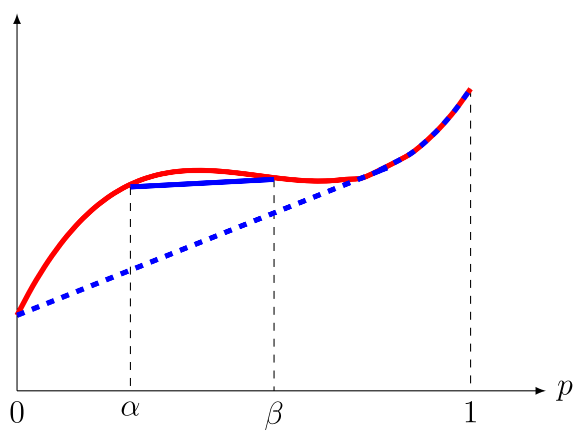

Figure 1: An illustration of convexification on different domains

The following result provides a characterization of the sender's worst case expected payoff from an information structure $\mu \in \mathcal{F}$ using the above notion of convexification.

Lemma 4. *For any* $\widehat{\Delta}$*,* $\pi \in \widehat{\Delta}$ *and* $\mu \in \mathcal{F}$*,*

$$V^\mu(\widehat{\Delta}, \pi) = \mathrm{co}_{\widehat{\Delta}}\phi^\mu(\pi).$$

Lemma 4 says that the worst case expected payoff to the sender from an information structure μ is equal to the value of the convexification of ϕ^μ over $\widehat{\Delta}$ at the initial prior π. This is analogous but opposite to the concavification result in Kamenica and

Gentzkow (2011). (See Corollary 2 in Kamenica and Gentzkow (2011) and Aumann et al. (1995).) To see why this is true, recall that $\phi^\mu(p)$ is the sender's expected payoff if the receiver's private belief is p. If the receiver's private belief is distributed according to some distribution $(\lambda^1 \circ p^1, \ldots, \lambda^K \circ p^K)$ where $K \geq 1$, $\lambda^1, \ldots, \lambda^K \geq 0$, $\sum_k \lambda^k = 1$, $p^1, \ldots, p^K \in \widehat{\Delta}$ and $\sum_k \lambda^k p^k = \pi$, then the sender's expected value is $\sum_k \lambda^k \phi^\mu(p^k)$. Because the sender does not know the exact distribution of the receiver's private belief, his worst case payoff is the lowest expected payoff over all such possible distributions of the receiver's private belief. As can be seen from the illustration in Figure 1, such lowest expected payoff precisely corresponds to the value of the convexification of ϕ^μ over $\widehat{\Delta}$ at π. The restriction of the domain to $\widehat{\Delta}$ reflects our assumption that the sender knows that the receiver's private belief is bounded in such set.

4 Value of persuasion

In this section, we study the sender's value of information design. Following Kamenica and Gentzkow (2011), we say that *the sender can benefit from persuasion* for the persuasion problem $(\widehat{\Delta}, \pi)$ if the sender can get a strictly higher payoff by designing an information structure than not supplying any information. Formally, define $\phi^0 : \Delta^{N-1} \to \mathbb{R}$ as

$$\phi^0(p) \equiv \sum_i p_i v\big(a[p], i\big), \ \forall p \in \Delta^{N-1}.$$

Notice that $\phi^0(p)$ is the sender's expected payoff if he does not supply any information and if the receiver's private belief is p. Similarly as before, for any $\widehat{\Delta}$, let $\mathrm{co}_{\widehat{\Delta}}\phi^0 : \widehat{\Delta} \to \mathbb{R}$ be the convexification of ϕ^0 over $\widehat{\Delta}$. By Lemma 4, the sender can benefit from persuasion for the persuasion problem $(\widehat{\Delta}, \pi)$ if and only if there exists $\mu \in \mathcal{F}$ such that $V^\mu(\widehat{\Delta}, \pi) > \mathrm{co}_{\widehat{\Delta}}\phi^0(\pi)$.

4.1 Full ambiguity

When $\widehat{\Delta} = \Delta^{N-1}$, the sender is completely uncertain about the receiver's private belief distribution because he thinks that the receiver can have any kind of private information structure. Among all possible private information structures is the one that perfectly reveals the underlying states to the receiver. If the receiver is indeed endowed with this information structure, it is obvious that no matter what information structure the sender chooses, he cannot change the receiver's behavior at all.

Consequently, because the sender evaluates each information structure by the worst case expected payoff and he thinks it is possible that the receiver's private information structure perfectly reveals the states, the sender's payoff from any information design cannot be higher than his expected payoff when the receiver perfectly observes the states. On the other hand, if the sender designs an information structure that perfectly reveals the underlying states, then the receiver will simply ignore her own private information and choose accordingly after knowing the states. This in turn implies that even if the sender does not know the true private information structure that the receiver has, the sender can always guarantee himself an expected payoff when the receiver perfectly observes the states, by fully revealing the underlying states to the receiver. As a result of the above analysis, if the sender has full ambiguity, fully revealing the states to the receiver is always optimal for the sender for any prior π. This is summarized in the following proposition. For each $i = 1, \dots, N$, let $\gamma^i \in \Delta^{N-1}$ be the belief that places probability 1 over states i.

Proposition 1. *If $\widehat{\Delta} = \Delta^{N-1}$, then full information disclosure is optimal for any prior $\pi \in \Delta^{N-1}$. In this case, the sender's value is*

$$V(\widehat{\Delta}, \pi) = \sum_{i=1}^{N} \pi_i v\big(a[\gamma^i], i\big), \ \forall \pi \in \Delta^{N-1}. \tag{11}$$

Another and perhaps a more enlightening way to understand Proposition 1 is through Lemma 4. For any persuasion problem $(\widehat{\Delta}, \pi)$, if *i)* the sender's expected payoff when the receiver's private belief is an extreme point of $\widehat{\Delta}$ is independent of his information design, and *ii)* there exists an information structure such that the sender's expected payoff from this information structure is a linear function of the receiver's private belief all over $\widehat{\Delta}$, then Lemma 4 directly implies that this information structure is optimal for the sender. Proposition 1 is a direct implication of this observation because the sender's expected payoff is independent of his information design when the receiver knows the underlying state and the sender's expected payoff is indeed a linear function in the receiver's private belief all over Δ^{N-1} (see (11)) when he discloses all information.

Although full information disclosure is optimal by Proposition 1, it may not be the unique optimal information structure for the sender. In particular, it is possible that the sender can get the same payoff from not supplying any information. This is the case, for example, if the receiver always chooses the sender's least-preferred action after observing the underlying state, i.e., $v\big(a[\gamma^i], i\big) = \min_{a \in A} v\big(a, i\big)$ for all $i = 1, \dots, N$. Thus, Proposition 1 alone does not tell us whether and when the sender can benefit from persuasion. The next proposition provides a complete answer to these questions.

Proposition 2. *Let $\widehat{\Delta} = \Delta^{N-1}$. Then the followings statements are equivalent:*

(i) There exists $\pi \in \Delta^{N-1}$, such that $\phi^0(\pi) < \sum_i \pi_i v\big(a[\gamma^i], i\big)$.

(ii) The sender benefits from persuasion for some $\pi \in \Delta^{N-1}$.

(iii) The sender benefits from persuasion for all $\pi \in \mathrm{int}\Delta^{N-1}$ where $\mathrm{int}\Delta^{N-1}$ is the set of all interior points of Δ^{N-1}.[14]

This proposition provides a full characterization of when the sender can benefit from fully revealing the states to the receiver if the sender has full ambiguity. Recall that $\phi^0(\pi) = \sum_i \pi_i v\big(a[\pi], i\big)$ is the sender's expected payoff from the receiver choosing her default action before observing any additional information. Proposition 2 thus, in short, tells us that the sender can indeed benefit from revealing the states to the receiver no matter what the interior initial prior is if and only if doing so makes him strictly better off than letting the receiver choose her default action *for some initial prior*. The "only if" part is straightforward: if for any prior, the sender cannot benefit from revealing information even if the receiver does not receive private signal, the sender can never gain from doing so if the receiver has private information and the sender cares about the worst case payoff. On the other hand, to understand the "if" part, suppose *(i)* holds for some π. Then any $\tilde{\pi}$ with full support is a convex combination of at most $N-1$ vertices of Δ^{N-1}, say $\gamma^1, \ldots, \gamma^{N-1}$, and π with strictly positive weight on π. In other words, if the initial prior is $\tilde{\pi}$, there exists an information structure that induces a distribution of the receiver's private beliefs over $\{\gamma^1, \ldots, \gamma^{N-1}, \pi\}$. Then *(i)* and simple algebra will show that the sender's expected payoff if he does not provide any information and if the receiver is endowed with the above private information structure is strictly less than $\sum_i \tilde{\pi}_i v\big(a[\gamma^i], i\big)$. This immediately implies that the worst case payoff to the sender at $\tilde{\pi}$ if he does not supply any information is strictly less than $\sum_i \tilde{\pi}_i v\big(a[\gamma^i], i\big)$. In other words, the sender can benefit from persuasion at $\tilde{\pi}$.

4.2 Local ambiguity

At the other end of the sender's ambiguity spectrum is the case in which the sender knows that the receiver only has little private information. We call this *local ambiguity*

[14] We use "interior" for short to mean the relative interior of Δ^{N-1}. In what follows, when we say an open set, we mean a relatively open set.

because the sender knows that the receiver's private information source can only provide coarse information and thus her possible private beliefs are all close to the common prior. Intuitively, when the sender faces local ambiguity, he can do almost as well as when the receiver does not have private information. This is indeed true for generic payoffs. Let $\widehat{A} = \bigcup_{p\in\Delta^{N-1}} \arg\max_{a\in A} \sum_i p_i u(a,i)$ be the set of all actions that maximize the receiver's expected payoff for some belief.

Proposition 3. *Suppose every $a \in \widehat{A}$ uniquely maximizes the receiver's expected payoff for some belief. Then for all $\pi \in \mathrm{int}\Delta^{N-1}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\widehat{\Delta} \subseteq O(\pi, \delta)$, $V(\widehat{\Delta}, \pi) > V(\{\pi\}, \pi) - \varepsilon$. As a result, for any $\pi \in \mathrm{int}\Delta^{N-1}$ and decreasing sequence of $\{\widehat{\Delta}_n\}_{n\geq 1}$ such that $\bigcap_n \widehat{\Delta}_n = \{\pi\}$, $\lim_n V(\widehat{\Delta}_n, \pi) = V(\{\pi\}, \pi)$.*

Although this proposition is intuitive, it is not straightforward. If the receiver does not have private information, then her private belief is just the common prior and the sender faces no ambiguity. In this case, we know that every information structure designed by the sender simply induces a distribution over the receiver's posterior beliefs which in turn induces a distribution over the receiver's actions. So does the sender's optimal information structure in this case. Now suppose the sender in fact faces local ambiguity. In this case, the receiver has coarse private information and so she may have private beliefs that are close but not exactly equal to the common prior. If the sender were to supply the information that maximizes his payoff with no ambiguity, it is true that this information structure will induce similar distributions over posterior beliefs regardless of the receiver's private beliefs. However, these similar distributions of posteriors may lead to vastly different actions by the receiver, as suggested by Proposition 5 in Kamenica and Gentzkow (2011), because the optimal information structure for no ambiguity under the common prior usually induces posterior beliefs at which the receiver is indifferent between several actions. So even a slight change in the receiver's private belief may result in a discrete change in the receiver's actions, which may yield very low payoff to the sender. In other words, optimal information structure for no ambiguity is usually not robust even to the receiver's local private beliefs.

The key idea behind Proposition 3 is that under the given assumption, there indeed exists a robust information structure that is close to the optimal one for no ambiguity. This information structure is robust because it not only induces similar distributions of posteriors, but also induces similar distributions over the receiver's actions as long as the receiver's private belief is close to the prior. Such information structure exists because the unique maximizer condition guarantees that each action that is possibly chosen by the receiver will be chosen not only for one particular posterior belief, but also for a wide range (an open set) of posterior beliefs. Because this information structure is close to the one that maximizes the sender's payoff when there is no ambiguity, it guarantees the sender a payoff close to $V(\{\pi\}, \pi)$ for all nearby private beliefs. Consequently, even the worst case payoff to the sender is guaranteed to be almost as high as $V(\{\pi\}, \pi)$. In Appendix B, we provide a simple example in which Proposition 3 fails because the unique maximizer condition is violated.

As a corollary of Proposition 3, we have

Proposition 4. *Suppose $\pi \in \text{int}\Delta^{N-1}$ and $V(\{\pi\}, \pi) > \phi^0(\pi)$.*[15] *Under the assumption of Proposition 3, there exists $\delta > 0$ such that the sender can benefit from persuasion for all persuasion problems $(\widehat{\Delta}, \pi)$ with $\widehat{\Delta} \subseteq O(\pi, \delta)$.*

5 A 2×2 example

In this section, we present a 2×2 example, where there are two states and the receiver only has two actions, to illustrate how we can apply the previous results to characterize the sender's optimal information design.

Suppose there are two states $i = 1$ and $i = 2$. The receiver has two actions $a = 1$ and $a = 2$. The receiver's ex post payoff is

$$u(a, i) = \begin{cases} 1, & \text{if } a = i, \\ 0, & \text{if } a \neq i. \end{cases}$$

So the receiver prefers taking the action that matches the underlying state.[16] We assume that the sender always prefers the receiver taking action $a = 2$ regardless of the underlying states. More specifically,

$$v(a, i) = \begin{cases} 1 & \text{if } a = 2, \\ 0 & \text{if } a = 1. \end{cases}$$

[15] Proposition 2 in Kamenica and Gentzkow (2011) gives both necessary and sufficient conditions on when this inequality holds.

[16] Because there are only two states and two actions, assuming that the receiver's ex post payoff is 0 when she chooses the wrong action entails no loss of generality. The assumption that the receiver's payoffs are the same in both states when she chooses the correct action is made purely for ease of exposition. The method we develop in this section can be easily extended to the case where the receiver gets different payoffs in different states when choosing the correct action.

Because there are only two states, we can identify agents' belief over the states by their belief $p \in [0,1]$ on state $i = 2$. Thus, the receiver's preferred action as a function of her posterior belief is[17]

$$a[p] = \begin{cases} 1, & \text{if } p < 1/2, \\ 2 & \text{if } p \geq 1/2. \end{cases}$$

Moreover, each probability distribution μ over $\Delta = \{(s_1, s_2) \in \mathbb{R}_+ | s_1 + s_2 = 1\}$ can be identified by a cumulative distribution function F over $[0,1]$ that represents the distribution of s_1. Thus, with slight abuse of notation, the set of all standard information structures over Δ can be written as

$$\mathcal{F} = \big\{\text{c.d.f } F \text{ over } [0,1] \,\big|\, \int_{[0,1]} s \mathrm{d}F(s) = 1/2\big\}.$$

When the receiver observes a signal $(s, 1-s)$ from a standard information structure and if her private belief is $(1-p, p)$, her posterior belief in (8) can be written as

$$\Big(\frac{(1-p)s}{(1-p)s + p(1-s)}, \frac{p(1-s)}{(1-p)s + p(1-s)}\Big).$$

Thus the receiver takes action $a = 2$ if and only if

$$\frac{p(1-s)}{(1-p)s + p(1-s)} \geq \frac{1}{2} \Longleftrightarrow s \leq p.$$

Therefore, for each standard information structure $F \in \mathcal{F}$, the sender's expected payoff from F when the receiver's private belief is p, i.e., $\phi^F(p)$ in (10), can be written as

$$\begin{aligned} \phi^F(p) &= \int_{[0,p]} \Big[(1-p)s + p(1-s)\Big] \mathrm{d}F(s) \\ &= pF(p) + (1-2p) \int_{[0,p]} s \mathrm{d}F(s). \end{aligned} \tag{12}$$

By Lemma 4, for any persuasion problem $([\alpha, \beta], \pi)$ where $0 \leq \alpha < \beta \leq 1$, the sender's worst case payoff from F is

$$V^F([\alpha, \beta], \pi) = \mathrm{co}_{[\alpha,\beta]} \phi^F(\pi),$$

and his optimal information design problem is

$$V([\alpha, \beta], \pi) = \max_{F \in \mathcal{F}} \; V^F([\alpha, \beta], \pi) = \max_{F \in \mathcal{F}} \; \mathrm{co}_{[\alpha,\beta]} \phi^F(\pi). \tag{13}$$

In the following subsections, we discuss the optimal solution to (13) for different cases of α and β.

[17] Recall that we assume that the receiver takes the sender's preferred action when the receiver is indifferent between the two actions.

5.1 $\alpha = 0$ and $\beta = 1$

We begin with the case where the sender has full ambiguity, to illustrate our results in Section 4.1. Figure 2 illustrates both Propositions 1 and 2. The solid red line represents ϕ^0. The solid blue line represents ϕ^F where

$$F(s) = \begin{cases} \frac{1}{2}, & \text{if } s \in [0,1), \\ 1 & \text{if } s = 1, \end{cases} \tag{14}$$

is the standard information structure that perfectly reveals the states to the receiver. To see why F is the sender's optimal information design, first notice that $\phi^F(0) = \phi^0(0) = 0$ and $\phi^F(1) = \phi^0(1) = \frac{1}{2}$ for all $F \in \mathcal{F}$ because no additional information can change the receiver's behavior if she already knows the underlying state. Therefore, by Lemma 4, we know for all $F \in \mathcal{F}$, $V^F([0,1],\pi) \leq (1-\pi)\times 0 + \pi \times \frac{1}{2} = \phi^F(\pi) = \text{co}_{[0,1]}\phi^F(\pi) = V^F([0,1],\pi)$. This immediately implies that F is optimal for all prior π and the sender's value is $V([0,1],\pi) = V^F([0,1],\pi)$. Moreover, because $\phi^0(\pi) < V([0,1],\pi)$ for all $\pi \in (0,1/2)$, Proposition 2 then implies that the sender can benefit from persuasion for all $\pi \in (0,1)$ by supplying full information. This can be directly verified by comparing ϕ^F and the blue dashed line in Figure 2, which is the convexification of ϕ^0 over $[0,1]$ and thus represents the sender's value if he does not supply any information. From the graph, we see that ϕ^F is higher than $\text{co}_{[0,1]}\phi^0$ for all interior π, implying that the sender can benefit from persuasion for all interior initial priors.

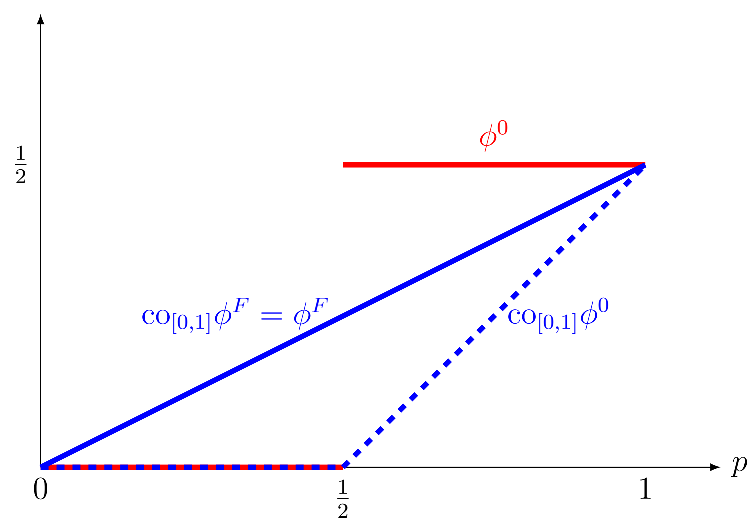

Figure 2: Full ambiguity

Another easy case is $\frac{1}{2} \leq \alpha < \beta \leq 1$ in which not supplying information is clearly optimal because doing so yields the highest expected payoff to the sender for all private beliefs $p \in [\alpha, \beta]$. In the next two sections, we consider the case $\alpha < \beta$ with $\alpha < \frac{1}{2}$ and illustrate how to apply Lemma 4 to characterize the sender's optimal information structures.

5.2 Linear-contingent-payoff information structures

In Section 4.1 we discussed that the characterization of optimal information design in the full ambiguity case (i.e., $\alpha = 0$ and $\beta = 1$) is simple mainly because there exists an information structure, i.e., F in (14), such that *i)* it simultaneously maximizes the sender's contingent payoff for both private beliefs $p = 0$ and $p = 1$, which are extreme points of $[0, 1]$, and *ii)* the sender's contingent payoff function ϕ^F is linear in the sender's private beliefs over the whole interval $[0, 1]$. The same logic also applies to the case $0 < \alpha < \frac{1}{2} < \beta = 1$. To see this, consider

$$F^{\alpha,\alpha}(s) = \begin{cases} 0 & \text{if } s \in [0, \alpha), \\ \frac{1}{2(1-\alpha)} & \text{if } s \in [\alpha, 1), \\ 1 & \text{if } s = 1. \end{cases} \tag{15}$$

By Kamenica and Gentzkow (2011), $F^{\alpha,\alpha}$ is the information structure that maximizes the sender's contingent payoff at α, i.e., $F^{\alpha,\alpha} = \arg\max_{F \in \mathcal{F}} \phi^F(\alpha)$. Because $\phi^F(1) = \frac{1}{2}$ for all $F \in \mathcal{F}$, $F^{\alpha,\alpha}$ trivially maximizes the sender's contingent payoff at 1. Moreover, it is easy to see from (12) that

$$\phi^{F^{\alpha,\alpha}}(p) = \frac{(1-2\alpha)p + \alpha}{2(1-\alpha)} = \frac{1-2\alpha}{2(1-\alpha)}(p - \alpha) + \alpha, \ \forall p \in [\alpha, 1].$$

Thus $\phi^{F^{\alpha,\alpha}}$ is linear in p over $[\alpha, 1]$. By the previous discussion, we immediately know that $F^{\alpha,\alpha}$ is optimal for every $\pi \in [\alpha, 1]$.

Intuitively, $F^{\alpha,\alpha}$ is no longer optimal for some $\pi \in (\alpha, \beta)$ when $\beta < 1$ because $F^{\alpha,\alpha}$ does not maximize the sender's contingent payoff when $p = \beta$. In fact, the characterization of the sender's optimal information structure becomes much more difficult when $\beta < 1$. However, as we will show, the crucial insight from the above analysis that for every prior $\pi \in [\alpha, 1]$, the sender's optimal value $V([\alpha, 1], \pi)$ is achieved by a linear contingent payoff function, i.e., $\phi^{F^{\alpha,\alpha}}$, is generalized to the case $\beta < 1$.

We say $F \in \mathcal{F}$ is a *linear-contingent-payoff information structure* over $[a, \beta]$ if $\phi^F(p) = \max\{0, \ell(p)\}$ over $[\alpha, \beta]$ for some increasing linear function ℓ. Obviously,

$F^{\alpha,\alpha}$ is a linear-contingent-payoff information structure over $[\alpha, \beta]$ for any $\beta > \alpha$. This concept generalizes $F^{\alpha,\alpha}$ and $\phi^{F^{\alpha,\alpha}}$ by allowing ϕ^F to take the form

$$\phi^F(p) = \begin{cases} 0 & \text{if } p \in [\alpha, x], \\ a(p-x) & \text{if } (x, \beta], \end{cases}$$

for some $x \in (\alpha, \beta)$ and $a > 0$.[18] Notice that if F is a linear-contingent-payoff information structure over $[\alpha, \beta]$, then $\text{co}_{[\alpha,\beta]}\phi^F = \phi^F|_{[\alpha,\beta]}$.

We now construct some candidates for the linear-contingent-payoff information structures. Fix $0 < \beta < 1$. For each pair $(x, b) \in \mathbb{R}^2$ where $0 < x < \min\{\frac{1}{2}, \beta\}$ and $0 \leq b \leq x$, let $F_\beta^{x,b} : [0,1] \to \mathbb{R}$ be the function defined as follows:

$$F_\beta^{x,b}(s) \equiv \begin{cases} 0, & \text{if } s \in [0, x), \\ (1-2x)a_\beta^{x,b} + 2b + \frac{(1-2x)b - 2x(1-x)a_\beta^{x,b}}{2\sqrt{x(1-x)}} \frac{1-2s}{\sqrt{s(1-s)}}, & \text{if } s \in [x, \beta), \\ (1-2x)a_\beta^{x,b} + 2b + \frac{(1-2x)b - 2x(1-x)a_\beta^{x,b}}{2\sqrt{x(1-x)}} \frac{1-2\beta}{\sqrt{\beta(1-\beta)}}, & \text{if } s \in [\beta, 1), \\ 1 & \text{if } s = 1. \end{cases} \tag{16}$$

where $a_\beta^{x,b} \in \mathbb{R}$ is the unique solution to the following linear equation:

$$2(1-x)\Big[1 - \sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\Big] a_\beta^{x,b} + \Big[2 + \frac{(1-2x)\sqrt{1-\beta}}{\sqrt{x(1-x)\beta}}\Big] b = 1. \tag{17}$$

These functions can be considered as generalizations of $F^{\alpha,\alpha}$ because $F_\beta^{\alpha,\alpha}$ (i.e., $x = b = \alpha$) defined by (16) simply degenerates to $F^{\alpha,\alpha}$ defined in (15). By continuity, let

$$F^{0,0}(s) \equiv \begin{cases} \frac{1}{2} & \text{if } s \in [0, 1), \\ 1 & \text{if } s = 1. \end{cases}$$

So $F^{0,0}$ is the standard information structure that perfectly reveals the underlying state in (14).

For general pair of (x, b), $F_\beta^{x,b}$ defined by (16) and (17) need not be a standard information structure. But we will show that with some further restrictions on (x, b), $F_\beta^{x,b}$ is guaranteed to be a linear-contingent-payoff standard information structure. For this, we first need a technical lemma.

[18] It is easy to show that F is a linear-contingent-payoff information structure if and only if ϕ^F is linear over $[\max\{\alpha, x\}, \beta]$ where $x = \min \text{supp} F$. In this sense, the terminology linear-contingent-payoff information structure is appropriate.

Lemma 5. *Suppose* $\frac{1}{2} < \beta < 1$. *For each* $x \in [1-\beta, \frac{1}{2})$, *there exists a unique* $b^*_\beta(x) \in [0, x]$ *such that* $F_\beta^{x, b^*_\beta(x)}(\beta) = 1$. *Moreover, the function* $b^*_\beta : [1-\beta, \frac{1}{2}) \to [0, \frac{1}{2})$ *is strictly increasing and onto.*

Based on Lemma 5, define

$$A(\alpha,\beta) \equiv \begin{cases} \{(\alpha,b) \big| 0 \le b \le \alpha\} \cup \{(x,0) \big| x \in (\alpha,\beta)\} & \text{if } \alpha < \beta \le \frac{1}{2}, \\ \{(\alpha,b) \big| 0 \le b \le \alpha\} \cup \{(x,0) \big| x \in (\alpha,1-\beta]\} & \text{if } \alpha < 1-\beta < \frac{1}{2} < \beta, \\ \{(\alpha,b) \big| b^*_\beta(\alpha) \le b \le \alpha\} & \text{if } 1-\beta \le \alpha < \frac{1}{2} < \beta, \end{cases}$$

The thick gray lines in Figures 3 - 5 illustrate the set $A(\alpha,\beta)$ for these three cases of α and β. The next lemma verifies that $F_\beta^{x,b}$ is a linear-contingent-payoff standard information structure for every pair of $(x,b) \in A(\alpha,\beta)$ and summarizes some important properties about $F_\beta^{x,b}$.

Lemma 6. *For all* $\alpha < \beta$ *and* $(x,b) \in A(\alpha,\beta)$, $F_\beta^{x,b}$ *satisfies the following properties:*

(i) $a_\beta^{x,b} > 0$.

(ii) $F_\beta^{x,b}$ *is a standard information structure.*

(iii) $\phi^{F_\beta^{x,b}}$ *satisfies*

$$\phi^{F_\beta^{x,b}}(p) = \begin{cases} 0, & \text{if } p \in [\alpha, x), \\ a^{x,b}(p-x) + b & \text{if } p \in [x, \beta]. \end{cases}$$

As a result, $\mathrm{co}_{[\alpha,\beta]}\phi^{F_\beta^{x,b}} = \phi^{F_\beta^{x,b}}|_{[\alpha,\beta]}$.

(iv) $(1-\beta)F_\beta^{x,b}(\beta) + \int_0^\beta F_\beta^{x,b}(s)\mathrm{d}s = 1/2$.

(v) If $\alpha < 1-\beta < \frac{1}{2} < \beta$ *and* $(x,b) = (1-\beta, 0) \in A(\alpha,\beta)$, *or* $1-\beta \le \alpha < \frac{1}{2} < \beta$ *and* $(x,b) = (\alpha, b^*_\beta(\alpha)) \in A(\alpha,\beta)$, *then* $\phi^{F_\beta^{x,b}}(\beta) = \frac{1}{2}$.

All these properties are easily verified from (16), (17) and the construction of $A(\alpha,\beta)$. Property *(ii)* states that $F_\beta^{x,b}$ is indeed a standard information structure for every $(x,b) \in A(\alpha,\beta)$. Properties *(iii)* and *(iv)* point out why these information structures are special. In particular, property *(iii)* states that $F_\beta^{x,b}$ is a linear-contingent-payoff information structure. The sender's expected payoff from each information structure $F_\beta^{x,b}$ with $(x,b) \in A(\alpha,\beta)$ over $[\alpha,\beta]$ is a nondecreasing and (piecewise) linear function whose convexification is itself. For example, consider the case where $\alpha < \beta < \frac{1}{2}$. When $x = \alpha$ and $b \in [0,\alpha]$, $\phi^{F_\beta^{\alpha,b}}$ is a linear function over $[\alpha,\beta]$ that passes through the point (α, b) with slope $a_\beta^{\alpha,b}$. When $x \in (\alpha, \beta)$, $\phi^{F_\beta^{x,0}}$ is a continuous piecewise linear function: it is 0 over $[\alpha, x]$ and a linear function over $[x, \beta]$ that passes through $(x, 0)$ with slope $a_\beta^{x,0}$. See Figure 3 for an illustration of this case. See also Figures 4 and 5 for the other two cases.

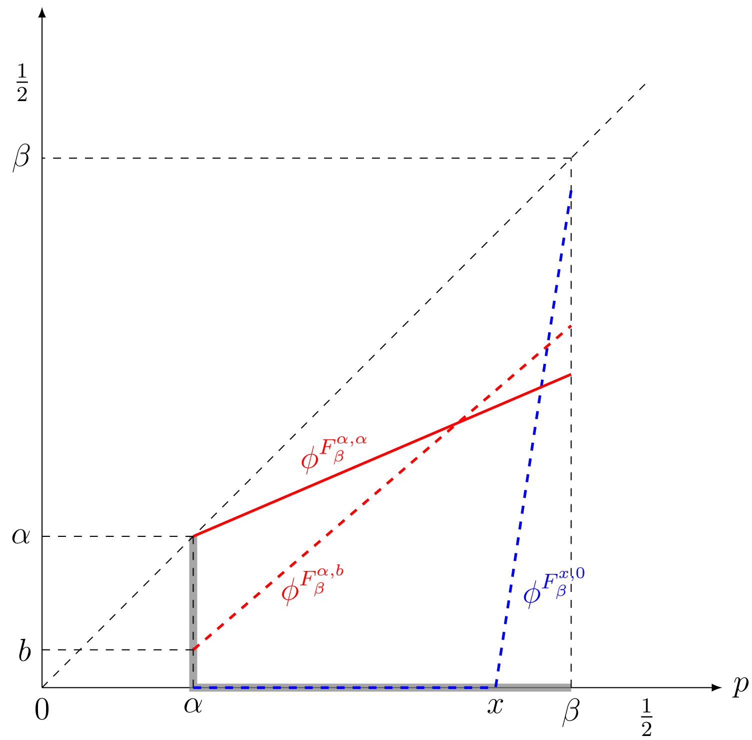

Figure 3: An illustration of $\phi^{F_\beta^{x,b}}$ when $\alpha < \beta < \frac{1}{2}$

Property *(iv)* is a simple implication of the facts that $F_\beta^{x,b}$ is a standard information structure and $F_\beta^{x,b}$ places no mass over the interval $(\beta, 1)$. To better understand this property, consider an arbitrary standard information structure $F \in \mathcal{F}$. We know

$$\begin{aligned}\frac{1}{2} &= \int_{[0,\beta]} s\mathrm{d}F(s) + \int_{(\beta,1]} s\mathrm{d}F(s) \\ &\leq \int_{[0,\beta]} s\mathrm{d}F(s) + (1 - F(\beta)) \\ &= 1 - (1-\beta)F(\beta) - \int_0^\beta s\mathrm{d}F(s),\end{aligned}$$

where the second equality comes from integration by parts.[19] Equivalently, we have

$$(1-\beta)F(\beta) + \int_0^\beta F(s)\mathrm{d}s \leq \frac{1}{2}. \tag{18}$$

[19]See for example Theorem 21.67 in Hewitt and Stromberg (1965) for integration by parts for Lebesgue-Stieltjes integrals.

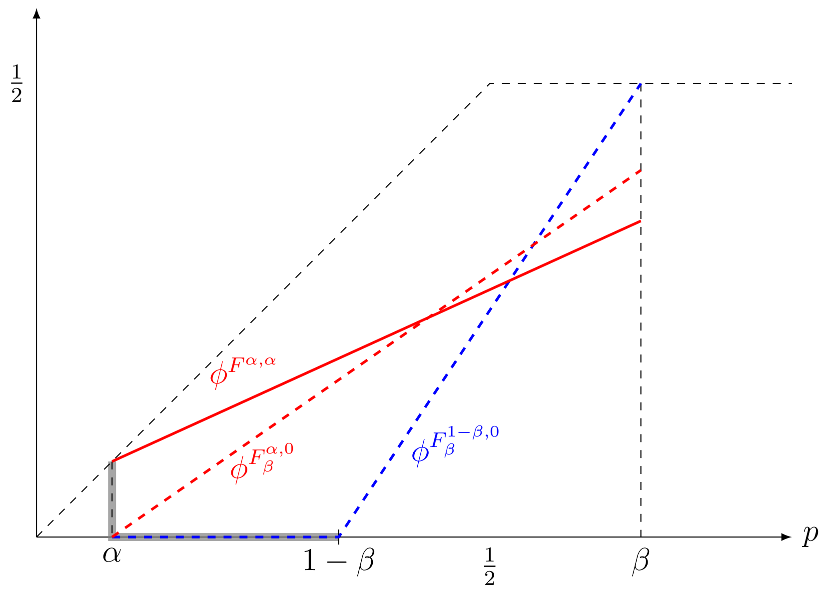

Figure 4: An illustration of $\phi^{F_\beta^{x,b}}$ when $\alpha < 1-\beta < \frac{1}{2} < \beta$

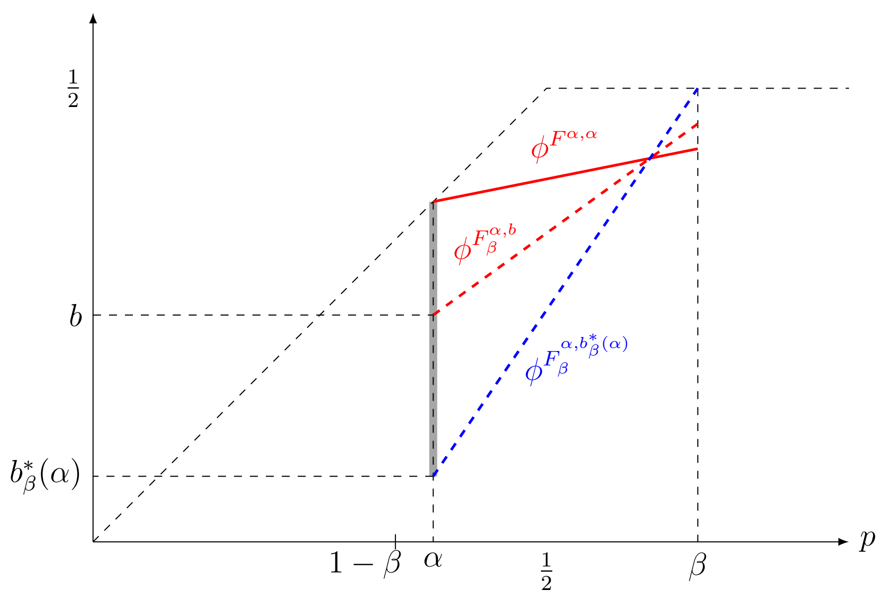

Figure 5: An illustration of $\phi^{F_\beta^{x,b}}$ when $1-\beta \leq \alpha < \frac{1}{2} < \beta$

Thus, the inequality (18) is a necessary condition for a distribution function over $[0,1]$ to be a standard information structure.[20] Property *(iv)* in Lemma 6 then simply states that $F_\beta^{x,b}$ satisfies (18) with equality. We will repeatedly use (18) in our characterization of optimal information structures.

Property (v) in Lemma 6, which is an implication of Lemma 5, simply states that when $\beta > \frac{1}{2}$, there exists a pair $(x,b) \in A(\alpha,\beta)$ such that $\phi^{F_\beta^{x,b}}$ achieves the sender's highest possible value at β.

5.3 Optimal information structures

With the preparations in Section 5.2, we are now ready to characterize the sender's optimal information structure when $0 \le \alpha < \beta < 1$ and $\alpha < \frac{1}{2}$.

Let $V^*_{\alpha,\beta} : (\alpha,\beta) \to \mathbb{R}$ be the upper envelope of all the conditional payoff functions in $\{\phi^{F_\beta^{x,b}}\}_{(x,b)\in A(\alpha,\beta)}$. That is,

$$V^*_{\alpha,\beta}(\pi) \equiv \max_{(x,b)\in A(\alpha,\beta)} \phi^{F_\beta^{x,b}}(\pi), \ \forall \pi \in (\alpha,\beta).$$

Because $F_\beta^{x,b}$ is a standard information structure for every pair $(x,b) \in A(\alpha,\beta)$ by Lemma 6, it is clear that $V([\alpha,\beta],\pi) \ge V^*_{\alpha,\beta}(\pi)$ for all $\pi \in (\alpha,\beta)$. The next proposition characterizes the sender's value function by showing that this inequality is in fact an equality. In other words, for every prior $\pi \in (\alpha,\beta)$, the sender's optimal value can be achieved by some linear-contingent-payoff information structure in $\{F_\beta^{x,b}\}_{(x,b)\in A(\alpha,\beta)}$.

Proposition 5. *Suppose* $0 \le \alpha < \beta < 1$ *and* $\alpha < \frac{1}{2}$. *For any* $F \in \mathcal{F}$, $\mathrm{co}_{[\alpha,\beta]}\phi^F(\pi) \le V^*_{\alpha,\beta}(\pi)$ *for all* $\pi \in (\alpha,\beta)$. *Therefore,* $V([\alpha,\beta],\pi) = V^*_{\alpha,\beta}(\pi)$ *for all* $\pi \in (\alpha,\beta)$.

Figures 7 - 9 illustrate the value function for different cases of α and β. The proof of this proposition is involved. The main idea is best understood by considering the case where $0 < \alpha < \beta \le \frac{1}{2}$. Consider any standard information structure $F \in \mathcal{F}$. For this moment, assume that *(i)* F places no mass over $[0,\alpha)$ and *(ii)* $\phi^F(p) = a(p-\alpha)+b$ for all $p \in [\alpha,\beta]$ for some $a > 0$ and $b \in [0,\alpha]$. Because $\phi^{F_\beta^{\alpha,b}}(p) = a_\beta^{\alpha,b}(p-\alpha)+b$ over $[\alpha,\beta]$ by Lemma 6, if we can show $a \le a_\beta^{\alpha,b}$, then we will have $\mathrm{co}_{[\alpha,\beta]}\phi^F = \phi^F \le \phi^{F_\beta^{\alpha,b}} \le V^*_{\alpha,\beta}$

[20]When $\beta \le \frac{1}{2}$, it is easy to prove that a nondecreasing and right continuous function $F : [0,\beta] \to [0,1]$ can be extended to a standard information structure if and only if (18) holds. In this sense, (18) is also sufficient. When $\beta > \frac{1}{2}$, then (18) and the condition $\beta - \int_0^\beta F(s)\mathrm{d}s < \frac{1}{2}$ together are both necessary and sufficient.

over (α, β) as desired. For this, first observe that using integration by parts, we can rewrite the sender's conditional payoff (12) as

$$\phi^F(p) = 2p(1-p)F(p) - (1-2p)\int_\alpha^p F(s)ds, \ \forall p \in [\alpha, \beta]. \tag{19}$$

The linearity of ϕ^F over $[\alpha, \beta]$ then implies

$$2p(1-p)F(p) - (1-2p)\int_\alpha^p F(s)\mathrm{d}s = a(p-\alpha) + b, \ \forall p \in [\alpha, \beta]. \tag{20}$$

This functional equation defines a differential equation

$$2p(1-p)y' - (1-2p)y = a(p-\alpha) + b \tag{21}$$

over $[\alpha, \beta]$ with the initial condition $y(\alpha) = 0$. By solving this differential equation, we can get

$$F(s) = (1-2\alpha)a + 2b + \frac{(1-2\alpha)b - 2\alpha(1-\alpha)a}{2\sqrt{\alpha(1-\alpha)}}\frac{1-2s}{\sqrt{s(1-s)}}, \ s \in [\alpha, \beta].$$

Notice that F and $F_\beta^{\alpha,b}$ over $[\alpha, \beta]$ only differ in a and $a_\beta^{\alpha,b}$. Because of property *(iv)* in Lemma (6), we can easily show, using the expression of F, that if $a > a_\beta^{\alpha,b}$, then we must have $2\beta(1-\beta)F(\beta) + \int_0^\beta F(s)\mathrm{d}s > 1/2$, which contradicts (18). Therefore $a \le a_\beta^{\alpha,b}$ and hence $\mathrm{co}_{[\alpha,\beta]}\phi^F = \phi^F \le \phi^{F_\beta^{\alpha,b}} \le V^*_{\alpha,\beta}$.

The real difficulties arise when we consider general F for which ϕ^F is not linear. In fact, ϕ^F is not even continuous and so the above approach does not directly apply. In dealing with these cases, we take two tricks. The first trick we take here is to approximate $\mathrm{co}_{[\alpha,\beta]}\phi^F$ by linear functions below ϕ^F. Thus, we only need to show that for any linear function l below ϕ^F over $[\alpha, \beta]$, there exists $F_\beta^{x,b}$ for some $(x, b) \in A(\alpha, \beta)$ such that $l \le \phi^{F_\beta^{x,b}}$. Consider a linear $l \le \phi^F$. Suppose again that $l(p) = a(p-\alpha) + b$ and assume $a > 0$ and $b \in [0, \alpha]$ for an illustration. As above, if we can show $a \le a^{\alpha,b}$, then $l \le \phi^{F_\beta^{\alpha,b}}$ as desired. However, notice that (19) implies that

$$2p(1-p)F(p) - (1-2p)\int_0^p F(s)\mathrm{d}s \ge a(p-\alpha) + b, \ \forall p \in [\alpha, \beta]. \tag{22}$$

Unlike (20) or (21), we now have a "differential inequality" (if F is continuous) instead of a differential equation. Obviously this cannot be explicitly "solved" as above. Our second trick is to show that the solution $G : [\alpha, \beta] \to \mathbb{R}$ to the differential equation

$$2p(1-p)G(p) - (1-2p)\int_\alpha^p G(s)\mathrm{d}s = a(p-\alpha) + b, \ \forall p \in [\alpha, \beta], \tag{23}$$

with the initial condition $\int_\alpha^\alpha G(s)\mathrm{d}s = 0$ is indeed everywhere below F over $[\alpha, \beta]$. This step is done by constructing a sequence of continuous functions $\{G_n\}_n$ over $[\alpha, \beta]$ with the property that $G_n \le F$ for all n and then showing that $\lim G_n = G$ by the contraction mapping theorem.[21] Based on this, we then can show as before that if $a > a_\beta^{\alpha,b}$, then G will violate (18) and so does F because $F \ge G$. Therefore, $l \le \phi^{F_\beta^{\alpha,b}}$. Of course, it is also possible that $b < 0$. In this case, there exists $x \in (\alpha, \beta)$ such that $l(x) = 0$. Then by a similar argument, we can show that $l \le \phi^{F_\beta^{x,0}}$. Figure 6 gives an illustration. Both l and l' are linear functions below ϕ^F and we show that $l \le \phi^{F_\beta^{\alpha,b}}$ and $l' \le \phi^{F_\beta^{x,0}}$. Finally, because every linear function below ϕ^F is bounded above by some $\phi^{F_\beta^{x,b}}$ and thus by $V^*_{\alpha,\beta}$, so is $\mathrm{co}_{[\alpha,\beta]}\phi^F$ as desired.

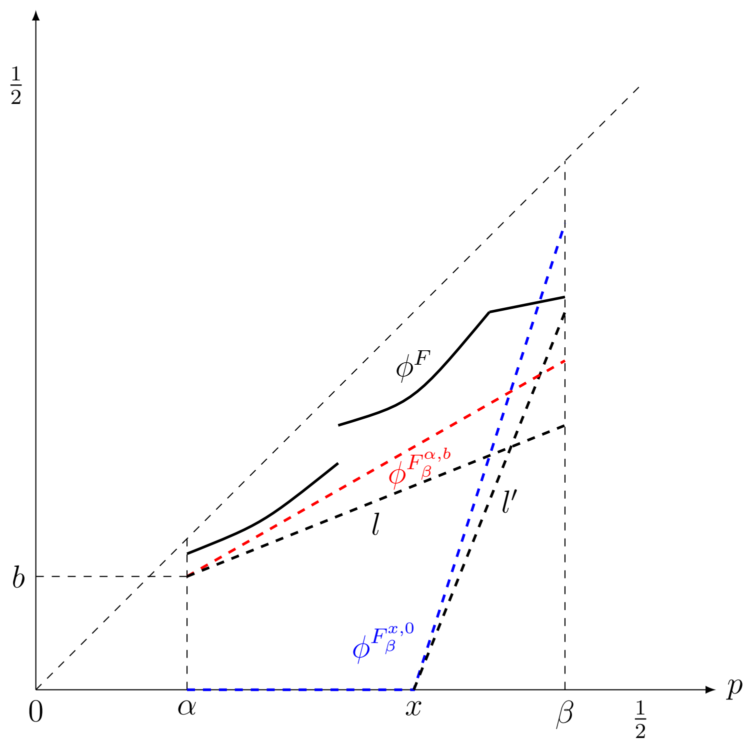

Figure 6: An illustration of the proof for Proposition 5

When $\beta > \frac{1}{2}$, the idea is similar as above: we show that $\mathrm{co}_{[\alpha,\beta]}\phi^F \le V^*_{\alpha,\beta}$ for every $F \in \mathcal{F}$ by showing that every linear function l below ϕ^F is bounded above by $V^*_{\alpha,\beta}$. However, another difficulty arises because when $\beta > \frac{1}{2}$, (22) no longer implies that F over $[\alpha, \beta]$ is above the solution G to (23).[22] The final trick we take here is to divide the

[21]The proof of this step is in the same spirit as proving the existence and uniqueness of the solution to a differential equation. See, for example Theorem 58.5 in Tenenbaum and Pollard (1985).

[22]This is due to two reasons. The first is that the term $1 - 2p$ as the coefficient of the term $\int_0^p F(s)\mathrm{d}s$ in (22) changes its sign as p increases from below 1/2 to above 1/2. The second is that

interval $[\alpha, \beta]$ into two subintervals $[\alpha, \frac{1}{2}]$ and $[\frac{1}{2}, \beta]$. If $\phi^F(p) \geq \ell(p) = a(p-x)+b$ over $[\alpha, \beta]$ for some $(x, b) \in A(\alpha, \beta)$, we show that F is bounded below by some function G_1 over $[\alpha, \frac{1}{2}]$ and another function G_2 over $[\frac{1}{2}, \beta]$.[23] We further show that if $a > a_\beta^{x,b}$, then $(1-\beta)G_2(\beta) + \int_\alpha^{\frac{1}{2}} G_1(s)\mathrm{d}s + \int_{\frac{1}{2}}^{\beta} G_2(s)\mathrm{d}s > \frac{1}{2}$. This would also imply that F violates (18) and hence $a \leq a_\beta^{x,b}$, which in turn implies $\ell \leq \phi_\beta^{F^{x,\beta}} \leq V^*_{\alpha,\beta}$.

Figures 7 - 9 illustrate the sender's optimal value function for different cases of α and β. Based on Proposition 5, the following proposition characterizes the sender's optimal information design.

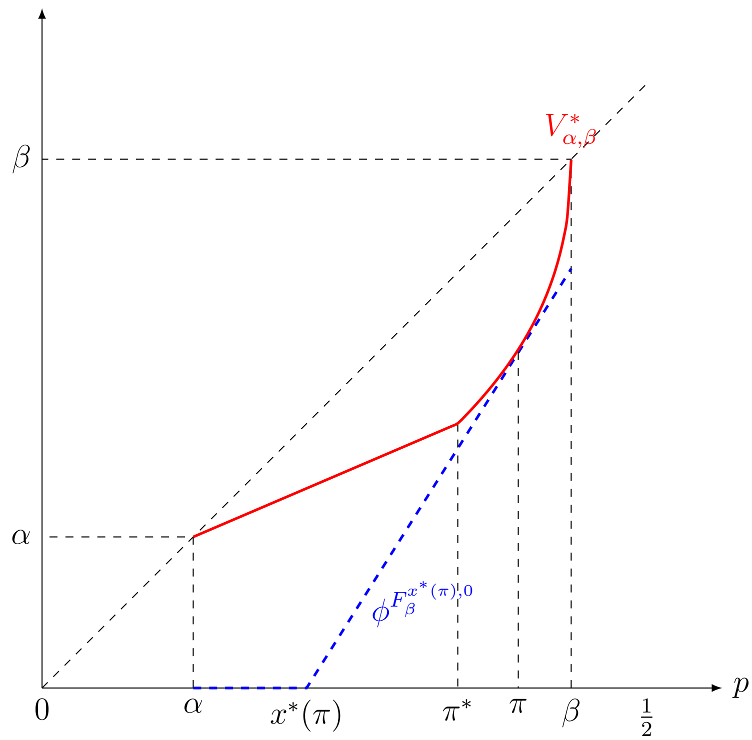

Figure 7: The value function when $\alpha < \beta < \frac{1}{2}$

Proposition 6. *Consider $0 \leq \alpha < \beta < 1$ and $\alpha < \frac{1}{2}$. Let*

$$\pi^*_{\alpha,\beta} \equiv \frac{2 - \sqrt{\frac{\alpha(1-\beta)}{(1-\alpha)\beta}}}{2 + \frac{1-2\alpha}{\alpha}\sqrt{\frac{\alpha(1-\beta)}{(1-\alpha)\beta}}} \in (\alpha, \beta) \tag{24}$$

if $\alpha > 0$ and $\pi^*_{\alpha,\beta} \equiv 0$ if $\alpha = 0$. For all prior $\pi \in (\alpha, \pi^*_{\alpha,\beta}]$, the sender's optimal information structure is $F^{\alpha,\alpha}$. For each $\pi \in (\pi^*_{\alpha,\beta}, \beta)$,

the contraction mapping we construct in previous case is no longer a contraction mapping when $\alpha < 1/2 < \beta$. For details, see Lemmas 9 - 11 in Appendix C.

[23]The function G_1 is similarly constructed as in the case with $\beta \leq \frac{1}{2}$. The construction of G_2 is more involved. See Lemma 11 for the construction.

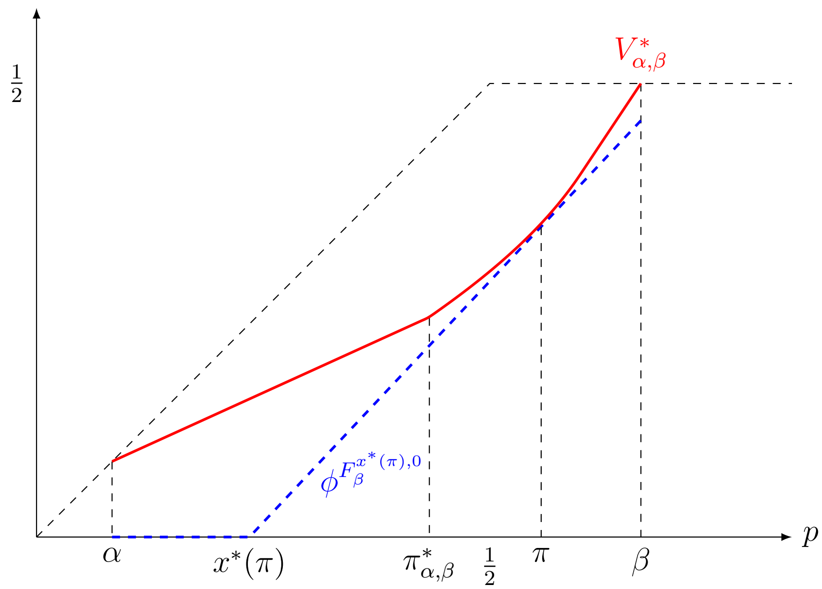

Figure 8: The value function when $\alpha < 1 - \beta < \frac{1}{2} < \beta$

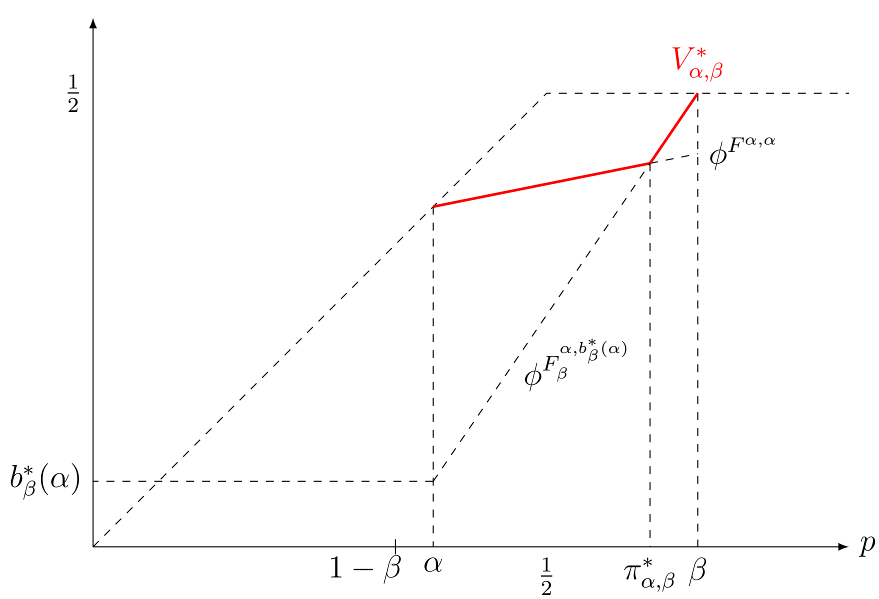

Figure 9: The value function when $1 - \beta < \alpha < \frac{1}{2} < \beta$

(i) $F_\beta^{x^*(\pi),0}$ *is optimal when* $\beta \le \frac{1}{2}$ *or* $\alpha < 1-\beta < \frac{1}{2} < \beta$*, where* $x^*(\pi)$ *is the unique solution to*

$$\max_{x\in(\alpha,\min\{\beta,1-\beta\})} a_\beta^{x,0}(\pi - x).$$

Moreover, x^* *is a strictly increasing function whose range is* $(\alpha, \min\{\beta, 1-\beta\})$*;*

(ii) $F_\beta^{\alpha,b^*_\beta(\alpha)}$ *is optimal when* $1-\beta \le \alpha < \frac{1}{2} < \beta$*.*

Proposition 6 states that the sender's optimal information structure is either $F^{\alpha,\alpha}$ or some information structure of the form $F_\beta^{x,b}$ (when $\alpha < \beta < \frac{1}{2}$ or $\alpha < 1-\beta < \frac{1}{2} < \beta$) or $F_\beta^{\alpha,b^*_\beta(\alpha)}$ (when $1-\beta < \alpha < \frac{1}{2} < \beta$), depending on the prior $\pi \in (\alpha, \beta)$. When the prior π is close to α, the sender's optimal information structure is exactly the one that maximizes his expected payoff when the prior is α and the receiver does not have private information. This is because when the prior is close to α, the receiver's private belief, no matter what private information structure the receiver has, must be close to α with large probability. Thus, to design an information structure that is robust to all possible private information structures, the sender must make sure that his information structure guarantees himself a high conditional payoff when the receiver's private belief is close to α. The information structure $F^{\alpha,\alpha}$ achieves this goal because it maximizes the sender's payoff at α. Notice that for fixed α, $\lim_{\beta\to 1} \pi^*_{\alpha,\beta} = 1$. In this case, $F^{\alpha,\alpha}$ is optimal for all $p \in [\alpha, 1]$. This coincides with our analysis for the case $\alpha < \frac{1}{2} < \beta = 1$ at the beginning of Section 5.2.

On the other hand, when the prior π is close to β, $F^{\alpha,\alpha}$ is no longer optimal. This is because in this case the receiver's private belief will be close to β rather than α with large probability regardless of her private information structure. Thus, in designing an optimal information structure, the sender actually cares more about his conditional payoff when the receiver's private belief is large. The information structures of the form $F_\beta^{x,b}$ serve this purpose when $\alpha < \beta < \frac{1}{2}$ or $\alpha < 1-\beta < \frac{1}{2}$. From Lemma 6, we know $\phi^{F_\beta^{x,b}}(p)$ is zero, which is the sender's lowest possible payoff, when $\alpha \le p \le x$. In other words, when the receiver's private belief falls in the range $[\alpha, x]$, $F_\beta^{x,b}$ does not change her behavior at all. The sender optimally gives up his opportunity to persuade the receiver when her private belief is very low just because this will only happen in rare cases no matter what the receiver's private information structure is. By choosing $F_\beta^{x,b}$, the sender makes sure that his conditional payoff is high as long as the receiver's private belief is around π and thus high, which occurs with large probability for all possible private information structures. When $1-\beta \le \alpha < \frac{1}{2}$, the sender no longer needs to give up his opportunity to persuade even when the receiver's private belief is around α. The information structure $F_\beta^{\alpha,b^*_\beta(\alpha)}$ achieves the sender's highest possible payoff when the receiver's private belief is β and thus guarantees the sender a high payoff when the receiver's private belief is close to β. But at the same time, $F_\beta^{\alpha,b^*_\beta(\alpha)}$ also yields a strictly positive payoff to the sender even when the receiver's private belief is close to α.

Information structures of the form $F_\beta^{\alpha,b}$ for $b \in [0, \alpha)$ when $\alpha < \beta < \frac{1}{2}$ or $\alpha < 1 - \beta < \frac{1}{2}$ and for $b \in (b^*_\beta(\alpha), \alpha)$ when $1 - \beta \leq \alpha < \frac{1}{2}$ do not appear in Proposition 6. These information structures are useful in simplifying the proof for Proposition 5, but they are never optimal except when $\pi = \pi^*_{\alpha,\beta}$. To understand this, simply observe that all the conditional payoff functions $\{\phi^{F_\beta^{\alpha,b}}\}_{(\alpha,b)\in A(\alpha,\beta)}$ intersect at $\pi^*_{\alpha,\beta} \in (\alpha, \beta)$. Consequently, they are dominated by $F^{\alpha,\alpha}$ when $\pi < \pi^*_{\alpha,\beta}$ and by $F^{\alpha,0}$ or $F_\beta^{\alpha,b^*_\beta(\alpha)}$ when $\pi > \pi^*_{\alpha,\beta}$. See, for example, Figure 5 for an illustration.

6 Conclusion

In this paper, we have studied robust Bayesian persuasion problems where the receiver has private information about which the sender only has limited knowledge. We provide a tractable framework and general methods that can be applied to various applications with more structure on preferences. The techniques we developed in characterizing optimal robust persuasion rules in the 2×2 example can also be applied to many other contexts outside the scope of Bayesian persuasion in which there are mean-restricted ambiguity and robustness concerns.

In this paper, we assume that the sender only uses public persuasion. That is, the sender designs a public information disclosure rule independent of the receiver's private information. Instead, like Kolotilin et al. (2016) and Bergemann et al. (forthcoming), we can think of environments in which the sender can use private persuasion. That is, the sender can condition information provided to the receiver on the receiver's reported type. In these environments, the sender must design a mechanism of private persuasion that is incentive compatible and robust to his knowledge about the distribution of the receiver's private beliefs. This is left for future research.

Appendix A Proofs for Section 3

Proof of Lemma 1. Consider any probability distribution ν over Δ^{N-1} that satisfies $\text{supp}(\nu) \subseteq \widehat{\Delta}$ and $\int_{\Delta^{N-1}} p\nu(\mathrm{d}p) = \pi$. Define $\nu_i \in \Delta(\Delta^{N-1})$ according to

$$\frac{\mathrm{d}\nu_i}{\mathrm{d}\nu}(p) = \frac{p_i}{\pi_i}, \ \forall p \in \Delta^{N-1}, \tag{25}$$

if $\pi_i > 0$ and let $\nu_i = \nu$ otherwise. If $\pi_i > 0$, we have $\nu_i(\Delta^{N-1}) = \int_{\Delta^{N-1}} (p_i/\pi_i)\nu(\mathrm{d}p) = \pi_i/\pi_i = 1$. Thus, $I = (\Delta^{N-1}, \nu_1, \ldots, \nu_N)$ is an information structure. From (25), it is easy to see that

$$q_i^I(\pi, p) = \begin{cases} \frac{p_i}{\sum_{j:\pi_j>0} p_j} & \text{if } \pi_i > 0, \\ 0 & \text{if } \pi_i = 0. \end{cases}$$

Since $\int_{\Delta^{N-1}} p_i \nu(\mathrm{d}p) = 0$ for all i such that $\pi_i = 0$, we know $\nu\big(\{p \in \Delta^{N-1} | p_i = 0 \text{ if } \pi_i = 0\}\big) = 1$. This in turn implies $q^I(\pi, p) = p$, ν-a.s. Thus, for any measurable $A \subset \Delta^{N-1}$,

$$\begin{aligned} \widehat{\nu}^\pi(A) &= \nu^\pi\big(q^I(\pi, \cdot) \in A\big) \\ &= \sum_{i=1}^N \pi_i \nu_i\big(q^I(\pi, \cdot) \in A\big) \\ &= \sum_{i:\pi_i>0} \pi_i \int_{q^I(\pi,\cdot)\in A} \frac{p_i}{\pi_i} \nu(\mathrm{d}p) \\ &= \int_A \nu(\mathrm{d}p), \end{aligned}$$

implying that $\widehat{\nu}^\pi = \nu$. Because $\text{supp}(\nu) \subseteq \widehat{\Delta}$, so is $\text{supp}(\widehat{\nu}^\pi)$. Therefore $I \in \widehat{\mathcal{I}}(\widehat{\Delta}, \pi)$, completing the proof. □

Proof of Lemma 2. Suppose $I_s = (S, \mu_1, \ldots, \mu_N)$ and $I'_s = (S', \mu'_1, \ldots, \mu'_N)$ are equivalent. Then for all $i = 1, \ldots, N$ and $p \in \Delta^{N-1}$,

$$\int_S v\big(a\big[q^{I_s}(p, s)\big], i\big)\mu_i(\mathrm{d}s) = \int_{S'} v\big(a\big[q^{I'_s}(p, s')\big], i\big)\mu'_i(\mathrm{d}s')$$

because I_s and I'_s always induce the same conditional posterior belief distribution. Therefore, for all $p \in \Delta^{N-1}$,

$$\sum_i p_i \int_S v\big(a\big[q^{I_s}(p, s)\big], i\big)\mu_i(\mathrm{d}s) = \sum_i p_i \int_{S'} v\big(a\big[q^{I'_s}(p, s')\big], i\big)\mu'_i(\mathrm{d}s'),$$

implying

$$\int_{\Delta^{N-1}} \Big(\sum_i p_i \int_S v\big(a\big[q^{I_s}(p,s)\big],i\big)\mu_i(\mathrm{d}s)\Big)\nu(\mathrm{d}p)$$

$$=\int_{\Delta^{N-1}} \Big(\sum_i p_i \int_{S'} v\big(a\big[q^{I'_s}(p,s')\big],i\big)\mu'_i(\mathrm{d}s')\Big)\nu(\mathrm{d}p)$$

for all $\nu \in \Delta(\Delta^{N-1})$. Hence $V^{I_s}(\widehat{\Delta},\pi) = V^{I'_s}(\widehat{\Delta},\pi)$ for all $\widehat{\Delta}$ and $\pi \in \widehat{\Delta}$ from (6). □

Proof of Lemma 3. In Blackwell (1951, 1953), two information structures are equivalent if the attainable payoffs from these two information structures are always the same for every decision problem. Then he showed that each information structure is equivalent to some standard information structure according to this notion. Although it is easy to show that this notion of equivalence is in fact equivalent to that in the current paper which in turn implies Lemma 3, here we provide a direct verification. The construction of the standard information structure below is copied from Blackwell (1951).

Consider any information structure $I = (S, \mu_1, \ldots, \mu_N)$. As in Blackwell (1951), define $f : S \to \Delta^{N-1}$ as

$$f(s) = \Big(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}(s), \ldots, \frac{\mathrm{d}\mu_N}{\mathrm{d}\mu_0}(s)\Big)/N,$$

and let $\mu_0^* \equiv \mu_0 \circ f^{-1}$. It is clear that μ_0^* is a measure over Δ^{N-1}. For each i, define μ_i^* by its Radon-Nikodym derivative $\mathrm{d}\mu_i^*/\mathrm{d}\mu_0^*(r)) = Nr_i$ for all $r \in \Delta^{N-1}$. It is easy to see that $\mu_0^* = (\sum_i \mu_i)/N$. Moreover, for all i,

$$\mu_i^*(\Delta^{N-1}) = \int_{\Delta^{N-1}} Nr_i\mu^*(\mathrm{d}r) = N\int_S f_i(s)\mu_0(\mathrm{d}s) = \int_S \mu_i(\mathrm{d}s) = 1.$$

Therefore, $(\Delta^{N-1}, \mu_1^*, \ldots, \mu_N^*)$ is a standard information structure. For any i, $p \in \Delta^{N-1}$ and any measurable set $A \subseteq \Delta^{N-1}$, we have

$$\begin{aligned}
&\int_{\{r\in\Delta^{N-1}|q(p,r)\in A\}} \mu_i^*(\mathrm{d}r)\\
=&N\int_{\{r\in\Delta^{N-1}|q(p,r)\in A\}} r_i\mu_0^*(\mathrm{d}r)\\
=&\int_{f^{-1}(\{r\in\Delta^{N-1}|q(p,r)\in A\})} \mu_i(\mathrm{d}s)\\
=&\int_{\{s\in S|q^I(p,s)\in A\}} \mu_i(\mathrm{d}s).
\end{aligned}$$

Hence I and $(\Delta^{N-1}, \mu_1^*, \ldots, \mu_N^*)$ are equivalent. □

Proof of Lemma 4. Observe that

$$V^{\mu}(\widehat{\Delta},\pi)=\inf_{\nu\in\mathcal{G}(\widehat{\Delta},\pi)}\int_{\widehat{\Delta}}\phi^{\mu}(p)\nu(\mathrm{d}p).$$

Because $\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}\leq\phi^{\mu}|_{\widehat{\Delta}}$ by definition, we know

$$V^{\mu}(\widehat{\Delta},\pi)\leq\inf_{\nu\in\mathcal{G}(\widehat{\Delta},\pi)}\int_{\widehat{\Delta}}\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}(p)\nu(\mathrm{d}p)\leq\inf_{\nu\in\mathcal{G}(\widehat{\Delta},\pi)}\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}\Big(\int_{\widehat{\Delta}}p\nu(\mathrm{d}p)\Big)=\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}(\pi),$$

where the second inequality comes from Jensen's inequality. On the other hand, because $\widehat{\Delta}$ is convex, it is well known that $\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}$ can be expressed as [24]

$$\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}(p)=\inf\left\{\sum_{k=1}^{N+1}\lambda^k p^k\;\middle|\;\begin{array}{l}p^k\in\widehat{\Delta},\ \lambda^k\geq 0\text{ for }k=1,\ldots,K,\\ \sum_{k=1}^{N+1}\lambda^k=1\text{ and }\sum_{k=1}^{N+1}\lambda^k p^k=p\end{array}\right\},\ \forall p\in\widehat{\Delta}.$$

Thus, it is immediate that $V^{\mu}(\widehat{\Delta},\pi)\leq\mathrm{co}_{\widehat{\Delta}}\phi^{\mu}(\pi)$, completing the proof. □

Appendix B Proofs for Section 4

Proof of Proposition 1. For any $\pi\in\Delta^{N-1}$, consider $\nu\in\Delta(\Delta^{N-1})$ such that ν puts probability π_i on γ^i. Obviously, $\nu\in\mathcal{G}(\Delta^{N-1},\pi)$. Therefore, we know for all $\mu\in\mathcal{F}$,

$$\begin{aligned}V^{\mu}(\Delta^{N-1},\pi)&\leq N\int_{\Delta^{N-1}}\Big(\int_{\Delta^{N-1}}\sum_i p_i s_i v\big(a\big[q(p,s)\big],i\big)\mu(\mathrm{d}s)\Big)\nu(\mathrm{d}p)\\&=N\int_{\Delta^{N-1}}\sum_i\pi_i s_i v\big(a[\gamma^i],i\big)\mu(\mathrm{d}s)\\&=\sum_i\pi_i v\big(a[\gamma^i],i\big),\end{aligned}$$

where the second equality comes from $\int_{\Delta^{N-1}}s_i\mu(\mathrm{d}\mu)=1/N$ for all i.

On the other hand, consider $\mu^*\in\mathcal{F}$ that puts probability $1/N$ on γ^i for all i. Notice that μ^* is the standard information structure that reveals all the information. Then, for any $p\in\Delta^{N-1}$,

$$\phi^{\mu^*}(\pi)=N\int_{\Delta^{N-1}}\sum_i p_i s_i v\big(a\big[q(p,s)\big],i\big)\mu^*(\mathrm{d}s)=\sum_i p_i v\big(a[\gamma^i],i\big).$$

Therefore,

$$V^{\mu^*}(\Delta^{N-1},\pi)=\mathrm{co}_{\Delta^{N-1}}\phi^{\mu^*}(\pi)=\sum_i\pi_i v\big(a[\gamma^i],i\big).$$

[24]See, for example, Corollary 17.1.5 in Rockafellar (1997).

It is then clear that

$$V(\Delta^{N-1}, \pi) = \sum_i \pi_i v\big(a[\gamma^i], i\big),$$

and thus μ^* is optimal for all π. $\square$

Proof of Proposition 2. $2 \implies 1$: Suppose, by contradiction, $\phi^0(\pi) \geq \sum_i \pi_i v\big(a[\gamma^i], i\big)$ for all π. Because $\pi \to \sum_i \pi_i v\big(a[\gamma^i], i\big)$ is linear and thus convex, we know $\mathrm{co}_{\widehat{\Delta}}\phi^0(\pi) \geq \sum_i \pi_i v\big(a[\gamma^i], i\big) = V(\widehat{\Delta}, \pi)$ for all π by the definition of convexification. This means that the sender cannot benefit from persuasion for any prior, a contradiction.

$3 \implies 2$: Obvious.

$1 \implies 3$: Suppose $\phi^0(\pi) < \sum_i \pi_i v\big(a[\gamma^i], i\big)$ for some π. Consider any $\tilde{\pi} \in \mathrm{int}\Delta^{N-1}$. Pick any j in $\arg\max_{i=1,\ldots,N} \pi_i/\tilde{\pi}_i$. Notice that we must have $\pi_j > 0$. Then for all $i \neq j$, we have $\tilde{\pi}_i - \tilde{\pi}_j\pi_i/\pi_j \geq 0$. This implies that $\tilde{\pi}$ is a convex combination of $\{\gamma^i\}_{i\neq j}$ and π, because

$$\tilde{\pi} = \sum_{i\neq j}\big(\tilde{\pi}_i - \frac{\tilde{\pi}_j}{\pi_j}\pi_i\big)\gamma^i + \frac{\tilde{\pi}_j}{\pi_j}\pi.$$

Therefore,

$$\begin{aligned}
\mathrm{co}_{\widehat{\Delta}}\phi^0(\tilde{\pi}) &\leq \sum_{i\neq j}\big(\tilde{\pi}_i - \frac{\tilde{\pi}_j}{\pi_j}\pi_i\big)\phi^0(\gamma^i) + \frac{\tilde{\pi}_j}{\pi_j}\phi^0(\pi) \\
&< \sum_{i\neq j}\big(\tilde{\pi}_i - \frac{\tilde{\pi}_j}{\pi_j}\pi_i\big)v\big(a[\gamma^i], i\big) + \frac{\tilde{\pi}_j}{\pi_j}\sum_i \pi_i v\big(a[\gamma^i], i\big) \\
&= \sum_i \tilde{\pi}_i v\big(a[\gamma^i], i\big) \\
&= V(\widehat{\Delta}, \pi).
\end{aligned}$$

implying that the sender benefits from persuasion for all interior priors. $\square$

To facilitate the proof for Proposition 3, we first need a technical lemma. Recall that the receiver's action choice mapping $\widehat{a} : \Delta^{N-1} \to A$ is a selection from the correspondence

$$p \mapsto \operatorname*{arg\,max}_{a\in \operatorname*{arg\,max}_{a'\in A}\sum_i p_i u(a',i)} \sum_i p_i v(a, i).$$

Clearly, the sender's payoff from information design does not depend on the particular selection of $\widehat{a}$. The next lemma states that we can always choose $\widehat{a}$ so that it has nice properties.

Lemma 7. *The receiver's action choice mapping $\widehat{a} : \Delta^{N-1} \to A$ can be chosen so that for all $a \in \widehat{a}[\Delta^{N-1}]$, $\widehat{a}^{-1}[a] \subseteq \Delta^{N-1}$ is convex.*

Proof. For ease of exposition, define $\Gamma_u : \Delta^{N-1} \rightrightarrows A$ as, for all $p \in \Delta^{N-1}$,

$$\Gamma_u(p) \equiv \arg\max_{a \in A} \sum_i p_i u(a, i).$$

Since A is finite, we can list A as $\{a^1, \ldots, a^K\}$ for some $K \geq 1$. For each $p \in \Delta^{N-1}$, let $\widehat{a}[p]$ be the action in $\arg\max_{a \in \Gamma_u(p)} \sum_i p_i v(a, i)$ with the lowest index. Suppose $\widehat{a}[p] = \widehat{a}[p'] = a^k$. We now argue that $\widehat{a}[p^\lambda] = a^k$ where $p^\lambda = \lambda p + (1-\lambda)p'$ for some $\lambda \in (0, 1)$.

Because $a^k \in \Gamma_u(p)$ and $a^k \in \Gamma_u(p')$, clearly we have $a^k \in \Gamma_u(p^\lambda)$. Suppose $a^l \in \Gamma_u(p^\lambda)$ for some $l \neq k$. Then the facts that $a^k \in \Gamma_u(p)$, $a^k \in \Gamma_u(p')$ and $a^l \in \Gamma_u(p^\lambda)$ together imply that $a^l \in \Gamma_u(p)$ and $a^l \in \Gamma_u(p')$. If $l < k$, then the fact that $\widehat{a}[p] = \widehat{a}[p'] = a^k$ and the construction of $\widehat{a}$ then immediately imply that $\sum_i p_i v(a^k, i) > \sum_i p_i v(a^l, i)$ and $\sum_i p'_i v(a^k, i) > \sum_i p'_i v(a^l, i)$, which in turn imply that $\sum_i p^\lambda_i v(a^k, i) > \sum_i p^\lambda_i v(a^l, i)$. Similarly, if $l > k$, we can show that $\sum_i p^\lambda v(a^k, i) \geq \sum_i p^\lambda v(a^l, i)$. Therefore, $\widehat{a}[p^\lambda] = a^k$. $\square$

Proof for Proposition 3. Let $\widehat{A} = \{a^1, \ldots, a^K\}$. Let $\widehat{a}$ be the one in Lemma 7. Because each a^k is the unique maximizer of the receiver's expected payoff for some belief, we have *(i)* $\widehat{a}[\Delta^{N-1}] = \widehat{A}$ and *(ii)* $\mathrm{int}\big(\widehat{a}^{-1}[a^k]\big) \neq \emptyset$ for $k = 1, \ldots, K$. Let $M = \max_{a \in A, 1 \leq i \leq N} |u(a, i)|$.

Fix $\varepsilon > 0$. Pick $\theta > 0$ such that $\theta < \min\{\frac{N\varepsilon}{10M}, 1\}$. Let $\tau_1 > 0$ be such that $O(\pi, \tau_1) \subseteq \Delta^{N-1}$. Because $\pi \in \mathrm{int}\Delta^{N-1}$, such that τ_1 exists. Pick $0 < \tau_2 < \tau_1$ such that $\tau_2/\tau_1 < \theta/(1-\theta)$.

Since $\mathcal{G}(\{\pi\}, \pi) = \{\nu \in \Delta(\Delta^{N-1}) | \nu(\{\pi\}) = 1\}$, clearly $V(\{\pi\}, \pi) = \max_{\mu \in \mathcal{F}} \phi^\mu(\pi)$. By Proposition 1 in Kamenica and Gentzkow (2011), $\max_{\mu \in \mathcal{F}} \phi^\mu(\pi)$ can be achieved by a standard information structure $\overline{\mu} \in \mathcal{F}$ such that $\mathrm{supp}\overline{\mu} \subseteq \{s^1, \ldots, s^K\}$ for some $s^1, \ldots, s^K \in \Delta^{N-1}$ that satisfy $\widehat{a}[q(\pi, s^k)] = a^k$ for $k = 1, \ldots, K$. For simplicity, let $q^k \equiv q(\pi, s^k)$ for $k = 1, \ldots, K$. So the unconditional distribution of posteriors induced by $\overline{\mu}$ given prior π can be written as $\lambda^1 \circ q^1 + \ldots + \lambda^K \circ q^K$ for some $(\lambda^1, \ldots, \lambda^K) \in \Delta^{N-1}$. Then

$$V(\{\pi\}, \pi) = \phi^{\overline{\mu}}(\pi) = \frac{1}{N} \sum_{k=1}^{K} \lambda^k \sum_{i=1}^{N} q^k_i v(a^k, i).$$

Pick $\kappa > 0$ such that for all $k = 1, \ldots, K$, $|\sum_{i=1}^N q^k_i v(a^k, i) - \sum_{i=1}^N q_i v(a^k, i)| < N\varepsilon/5$ for all $q \in O(q^k, \kappa)$. For each $k = 1, \ldots, K$, because $\widehat{a}^{-1}[a^k]$ is convex and has nonempty interior, it is well known that every point in $\widehat{a}^{-1}[a^k]$ can be approximated by points in $\mathrm{int}\big(\widehat{a}^{-1}[a^k]\big)$.[25] Thus, for each $k = 1, \ldots, K$, we can pick

[25] See, for example, Theorem 6.3 in Rockafellar (1997).

$\widetilde{q}^k \in O(q^k, \min\{\tau_2, \kappa\})$ such that $\widetilde{q}^k \in \operatorname{int}\big(\widehat{a}^{-1}[a^k]\big)$. Because $|\widetilde{q}^k - q^k| < \tau_2$ for all k, we know $\sum_k \lambda^k \widetilde{q}^k \in O(\pi, \tau_2)$ since $\sum_k \lambda^k q^k = \pi$. Then there exists a $\widetilde{q}^{K+1}$ such that $|\widetilde{q}^{k+1} - \pi| = \tau_1$ and π is a convex combination of $\sum_k \lambda^k \widetilde{q}^k$ and $\widetilde{q}^{K+1}$. That is, there exists $\widetilde{\theta} \in [0,1]$ such that

$$(1-\widetilde{\theta})\sum_{k=1}^{K} \lambda^k \widetilde{q}^k + \widetilde{\theta}\widetilde{q}^{K+1} = \pi. \tag{26}$$

Notice that $(1-\widetilde{\theta})|\pi - \sum_k \lambda^k \widetilde{q}^k| = \widetilde{\theta}|\pi - \widetilde{q}^{K+1}|$ implies

$$\frac{\widetilde{\theta}}{1-\widetilde{\theta}} < \frac{\tau_2}{\tau_1} < \frac{\theta}{1-\theta},$$

or equivalently $\widetilde{\theta} < \theta$.

For each $k = 1, \ldots, K+1$, define

$$\widetilde{s}^k \equiv \left(\frac{\widetilde{q}^k_1/\pi_1}{\sum_{j=1^N} \widetilde{q}^k_j/\pi_j}, \ldots, \frac{\widetilde{q}^k_N/\pi_N}{\sum_{j=1^N} \widetilde{q}^k_j/\pi_j} \right) \in \Delta^{N-1}.$$

Define $\widetilde{\mu} \in \Delta(\{\widetilde{s}^1, \ldots, \widetilde{s}^{K+1}\})$ such that $\widetilde{\mu}(\{\widetilde{s}^k\}) = \frac{\widetilde{\lambda}^k}{N}\sum_j \frac{\widetilde{q}^k_j}{\pi_j}$ for $k = 1, \ldots, K+1$, where $\widetilde{\lambda}^k = (1-\widetilde{\theta})\lambda^k$ if $k = 1, \ldots, K$ and $\widetilde{\lambda}^{K+1} = \widetilde{\theta}$. It is easy to verify that *(i)* $q(\pi, \widetilde{s}^k) = \widetilde{q}^k$ for $k = 1, \ldots, K+1$, *(ii)* $\widetilde{\mu}$ is a standard information structure, and *(iii)* the unconditional distribution of posterior beliefs induced by $\widetilde{\mu}$ given prior π is $\widetilde{\lambda}^1 \circ \widetilde{q}^1 + \ldots + \widetilde{\lambda}^{K+1} \circ \widetilde{q}^{K+1}$. It is then easy to see that for all $p \in \Delta^{N-1}$.

$$\phi^{\widetilde{\mu}}(p) = \frac{1}{N}\sum_{k=1}^{K+1} \widetilde{\lambda}^k \sum_{i=1}^{N} \frac{p_i \widetilde{q}^k_i}{\pi_i} v\big(a[q(p, \widetilde{s}^k)], i\big).$$

Specifically, when $p = \pi$, we have

$$\phi^{\widetilde{\mu}}(\pi) = \frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K} \lambda^k \sum_{i=1}^{N} \widetilde{q}^k_i v(a^k, i) + \frac{\widetilde{\theta}}{N}\sum_{i=1}^{N} \widetilde{q}^{K+1}_i v\big(a[\widetilde{q}^{K+1}], i\big)$$

Because $|\widetilde{q}^k - q^k| < \kappa$ for all $k = 1, \ldots, K$ by construction, we know

$$\begin{aligned}
&\left| \frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K} \lambda^k \sum_{i=1}^{N} \widetilde{q}^k_i v(a^k, i) - \frac{1-\widetilde{\theta}}{N}\phi^{\overline{\mu}}(\pi) \right| \\
=&\frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K} \lambda^k \left| \sum_{i=1}^{N} \widetilde{q}^k_i v(a^k, i) - \sum_{i=1}^{N} q^k_i v(a^k, i) \right| \\
<&\frac{1}{N}\frac{N\varepsilon}{5}\sum_{k=1}^{K} \lambda^k \\
=&\frac{\varepsilon}{5}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\widetilde{q}_i^{K+1}v\big(a[\widetilde{q}^{K+1}],i\big)-\frac{\widetilde{\theta}}{N}\phi^{\overline{\mu}}(\pi)\right|\\
\leq&\frac{\widetilde{\theta}}{N}\sum_{k=1}^{K}\lambda^k\left|\sum_{i=1}^{N}\widetilde{q}_i^{K+1}v\big(a[\widetilde{q}^{K+1}],i\big)-\sum_{i=1}^{N}q_i^kv(a^k,i)\right|\\
\leq&\frac{\widetilde{\theta}}{N}\sum_{k=1}^{K}2M\lambda^k\\
\leq&\frac{2\theta M}{N}\\
<&\frac{\varepsilon}{5}.
\end{aligned}$$

Therefore, we know $\phi^{\widetilde{\mu}}(\pi)>\phi^{\overline{\mu}}(\pi)-2\varepsilon/5$.

Because for all $k=1,\dots,K$, $\widetilde{q}^k$ is in the interior of $\widehat{a}^{-1}[a^k]$ and $q(\pi,\widetilde{s}^k)=\widetilde{q}^k$ by construction, and because $q(\cdot,s):\Delta^{N-1}\to\Delta^{N-1}$ is continuous, there exists $\delta_1>0$ such that for all $p\in O(\pi,\delta_1)$, $a\big[q(p,\widetilde{s}^k)\big]=a^k$ for $k=1,\dots,K$. Therefore, for $p\in O(\pi,\delta_1)$, $\phi^{\widetilde{\mu}}(p)$ can be rewritten as

$$\phi^{\widetilde{\mu}}(p)=\frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K}\lambda^k\sum_{i=1}^{N}\frac{p_i\widetilde{q}_i^k}{\pi_i}v(a^k,i)+\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\frac{p_i\widetilde{q}_i^{K+1}}{\pi_i}v\big(a[q(p,\widetilde{s}^{K+1})],i\big)$$

Because the first term is linear in p, we know there exists a $\delta_2>0$ such that for all $p\in O(\pi,\delta_2)$,

$$\left|\frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K}\lambda^k\sum_{i=1}^{N}\frac{p_i\widetilde{q}_i^k}{\pi_i}v(a^k,i)-\frac{1-\widetilde{\theta}}{N}\sum_{k=1}^{K}\lambda^k\sum_{i=1}^{N}\widetilde{q}_i^kv(a^k,i)\right|\leq\frac{\varepsilon}{5}.$$

Moreover, when $p\in O(\pi,2\min_i\pi_i)$, we have

$$\begin{aligned}
&\left|\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\frac{p_i\widetilde{q}_i^{K+1}}{\pi_i}v\big(a[q(p,\widetilde{s}^{K+1})],i\big)-\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\widetilde{q}_i^{K+1}v\big(a[\widetilde{q}^{K+1}],i\big)\right|\\
\leq&\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\left|\frac{p_i-\pi_i}{\pi_i}\right|\widetilde{q}_i^{K+1}\left|v\big(a[q(p,\widetilde{s}^{K+1})],i\big)\right|\\
&+\frac{\widetilde{\theta}}{N}\sum_{i=1}^{N}\widetilde{q}_i^{K+1}\left|v\big(a[q(p,\widetilde{s}^{K+1})],i\big)-v\big(a[\widetilde{q}^{K+1}],i\big)\right|\\
\leq&\frac{2M\widetilde{\theta}}{N}+\frac{2M\widetilde{\theta}}{N}\\
\leq&\frac{4M\theta}{N}\\
<&\frac{2\varepsilon}{5}.
\end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2, 2\min_i \pi_i\} > 0$. From the above analysis, we know that for all $p \in O(\pi, \delta)$, $\phi^{\widetilde{\mu}}(p) > \phi^{\widetilde{\mu}}(\pi) - 3\varepsilon/5 > \phi^{\overline{\mu}}(\pi) - \varepsilon$. Therefore, for all $\widehat{\Delta} \subseteq O(\pi, \delta)$, we have

$$V(\widehat{\Delta}, \pi) \geq \max_{\mu \in \mathcal{F}} \inf_{p \in \widehat{\Delta}} \phi^{\mu}(p) \geq \inf_{p \in \widehat{\Delta}} \phi^{\widetilde{\mu}}(p) \geq \phi^{\overline{\mu}}(\pi) - \varepsilon = V(\{\pi\}, \pi) - \varepsilon,$$

where the first inequality comes from Lemma 4 and the fact that the constant mapping $p' \mapsto \inf_{p \in \widehat{\Delta}} \phi^{\mu}(p)$ over $\widehat{\Delta}$ is a convex function below ϕ^{μ}. This completes the proof. □

Proof for Proposition 4. Let $\varepsilon = V(\{\pi\}, \pi) - \phi^0(\pi) > 0$. By Proposition 3, there exists $\delta > 0$ such that for all $\widehat{\Delta} \subset O(\pi, \delta)$, $V(\widehat{\Delta}, \pi) > V(\{\pi\}, \pi) - \varepsilon = \phi^0(\pi) \geq \text{co}_{\widehat{\Delta}} \phi^0(\pi)$. □

An example. We present a simple example in which Proposition 3 fails because the unique maximizer condition does not hold for nongeneric payoffs.

Assume there are two states $i = 1, 2$ and three actions $A = \{a_1, a_2, a_3\}$. Assume the receiver strictly prefers a_1 if her belief over state $i = 2$ is less than $1/2$, a_2 if her belief is greater than $1/2$ and is indifferent between all actions when belief is exactly $1/2$. Assume the sender's payoff is 0 if the receiver chooses either a_1 or a_2 regardless of the states and 1 if the receiver chooses a_3. So in any sender preferred subgame perfect equilibrium, $\widehat{a}[p] = a_1$ if $p < 1/2$, $\widehat{a}[p] = a_2$ if $p > 1/2$ and $\widehat{a}[p] = a_3$ if $p = 1/2$. Notice that a_3 is never the unique maximizer of the receiver's expected payoff.

It is easy to see that $V(\{\pi\}, \pi) > 0$ for all $\pi \in (0, 1)$. However, for any $\alpha < \pi < \beta$, $V([\alpha, \beta], \pi) = 0$. To see this, consider an arbitrary information structure $\mu \in \mathcal{F}$. Notice that if $\phi^{\mu}(p) > 0$ for some $0 < p < 1$, then μ must put strictly positive probability over the signal $(p, 1-p)$. Because μ can have at most countably many atoms, there must exist $p_1 \in (\alpha, \pi)$ and $p_2 \in (\pi, \beta)$ such that $\mu(\{(p_1, 1-p_1)\}) = \mu(\{(p_2, 1-p_2)\}) = 0$. This implies that $\phi^{\mu}(p_1) = \phi^{\mu}(p_2) = 0$ and hence $\text{co}_{[\alpha,\beta]}\phi^{\mu}(\pi) = 0$. □

Appendix C Proofs for Section 5

Proof for Lemma 5. Suppose $1-\beta \le x < \frac{1}{2} < \beta < 1$. Consider the following system of linear equations in (a,b):

$$\sqrt{x(1-x)}\left[\frac{1-2x}{\sqrt{x(1-x)}} - \frac{1-2\beta}{\sqrt{\beta(1-\beta)}}\right]a + \left[2 + \frac{(1-2x)(1-2\beta)}{2\sqrt{x(1-x)}\sqrt{\beta(1-\beta)}}\right]b = 1, \tag{27}$$

$$2(1-x)\left[1-\sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right]a + \left[2 + \frac{(1-2x)\sqrt{1-\beta}}{\sqrt{x(1-x)\beta}}\right]b = 1. \tag{28}$$

Notice that (28) is simply the defining equation of $a_\beta^{x,b}$ in (17) and given (28), (27) is simplify the condition that $F_\beta^{x,b}(\beta) = 1$. When $1-\beta \le x < \frac{1}{2}$, this system of linear equations has a unique solution

$$b_\beta^*(x) = \frac{\frac{1-x}{1-\beta} - \sqrt{\frac{(1-x)\beta}{x(1-\beta)}}}{\left[\sqrt{\frac{(1-x)\beta}{x(1-\beta)}} - 1\right]^2}.$$

Then it is straightforward to verify that $0 \le b_\beta^*(x) \le x$. Moreover, simple algebra will show that $b_\beta^*(\cdot) : [1-\beta, \frac{1}{2}] \to [0, \frac{1}{2}]$ is strictly increasing, $b_\beta^*(1-\beta) = 0$ and $b_\beta^*(\frac{1}{2}) = \frac{1}{2}$. Thus $b_\beta^*(\cdot)$ is onto.

□

Proof for Lemma 6. Consider $\alpha < \beta < 1$ and $(x,b) \in A(\alpha,\beta)$. Notice that we always have $0 \le b \le x < \frac{1}{2}$ and $x < \beta$.

From (17), we know

$$a_\beta^{x,b} = \frac{1 - \left[2 + \frac{(1-2x)\sqrt{1-\beta}}{\sqrt{x(1-x)\beta}}\right]b}{2(1-x)\left[1-\sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right]} \ge \frac{1 - \left[2 + \frac{(1-2x)\sqrt{1-\beta}}{\sqrt{x(1-x)\beta}}\right]x}{2(1-x)\left[1-\sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right]} = \frac{1-2x}{2(1-x)} > 0.$$

So *(i)* holds.

For *(ii)*, first observe that, by (17),

$$(1-2x)b - 2x(1-x)a_\beta^{x,b} = \frac{b-x}{1-\sqrt{\frac{x(1-\beta)}{(1-x)\beta}}} \le 0.$$

Because the mapping $s \mapsto (1-2s)/\sqrt{s(1-s)}$ strictly decreases over $(0,1)$, we know $F_\beta^{x,b}$ is nondecreasing over $[x,\beta]$. Using (16) and (17), it is easy to see that

$$F_\beta^{x,b}(x) = \frac{b}{2x(1-x)} \ge 0.$$

Moreover, because

$$F_\beta^{x,b}(\beta) = \frac{1}{2} + \frac{x\sqrt{(1-x)\beta} + (\sqrt{x(1-\beta)} - \sqrt{(1-x)\beta})b}{2(1-x)\sqrt{x(1-\beta)}} \tag{29}$$

and because $x < \beta$, we know $F_\beta^{x,b}(\beta)$ decreases in b. Thus, when $\alpha < \beta \leq \frac{1}{2}$ or $\alpha < 1-\beta < \frac{1}{2}$, we know

$$F_\beta^{x,b}(\beta) \leq \frac{1}{2} + \frac{x\sqrt{(1-x)\beta}}{2(1-x)\sqrt{x(1-\beta)}} = \frac{1}{2} + \frac{\sqrt{x\beta}}{2\sqrt{(1-x)(1-\beta)}} \leq 1,$$

where the second inequality comes from either $x < \beta \leq 1/2$ or $x \leq 1-\beta < \frac{1}{2}$. When $1-\beta \leq \alpha < \frac{1}{2}$, (29) and Lemma 5 directly imply that $F_\beta^{\alpha,b}(\beta) \leq F_\beta^{\alpha,b^*_\beta(\alpha)}(\beta) = 1$. Therefore, in all cases $F_\beta^{x,b}$ is nondecreasing over $[0,1]$. Since it is obvious that $F_\beta^{x,b}$ is right continuous, we know $F_\beta^{x,b}$ is a distribution function over $[0,1]$. Finally,

$$\begin{aligned}
\int_{[0,1]} s\mathrm{d}F_\beta^{x,b}(s) &= \int_{[x,\beta]} s\mathrm{d}F_\beta^{x,b}(s) + (1 - F_\beta^{x,b}(\beta)) \\
&= \beta F_\beta^{x,b}(\beta) - \int_x^\beta F_\beta^{x,b}(s)\mathrm{d}s + (1 - F_\beta^{x,b}(\beta)) \\
&= 1 - (1-x)\left[1 - \sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right] a_\beta^{x,b} - \left[1 + \frac{(1-2x)\sqrt{1-\beta}}{2\sqrt{x(1-x)\beta}}\right] b \\
&= \frac{1}{2},
\end{aligned}$$

where the second inequality comes from integration by parts. This proves that $F_\beta^{x,b}$ is a standard information structure. Notice that the second equality also proves *(iv)*.

For *(iii)*, first observe that the fact that $F_\beta^{x,b}(s) = 0$ for all $s \in [0,x)$ implies $\phi^{F_\beta^{x,b}}(p) = 0$ for all $p < x$ by (12). For $p \in [x,\beta]$, from (12) again, we have

$$\begin{aligned}
\phi^{F_\beta^{x,b}}(p) &= 2p(1-p)F_\beta^{x,b}(p) - (1-2p)\int_x^p F_\beta^{x,b}(s)\mathrm{d}s \\
&= 2p(1-p)\left[(1-2x)a_\beta^{x,b} + 2b + \frac{(1-2x)b - 2x(1-x)a_\beta^{x,b}}{2\sqrt{x(1-x)}}\frac{1-2p}{\sqrt{p(1-p)}}\right] \\
&\quad - (1-2p)\Bigg[\Big((1-2x)a_\beta^{x,b} + 2b\Big)(p-x) + \frac{(1-2x)b - 2x(1-x)a_\beta^{x,b}}{\sqrt{x(1-x)}} \\
&\qquad \times \left(\sqrt{p(1-p)} - \sqrt{x(1-x)}\right)\Bigg] \\
&= a_\beta^{x,b}(p-x) + b,
\end{aligned}$$

proving *(iii)*.

For *(v)*, let $(x,b)=(1-\beta,0)$ if $\alpha<1-\beta<\frac{1}{2}$ and $(x,b)=(\alpha,b^*_\beta(\alpha))$. We know $F^{x,b}_\beta(\beta)=1$ for the first case by (29) and for the second case by Lemma 5. Therefore, by *(ii)* in the previous steps, we have

$$\phi^{F^{x,b}_\beta}(\beta)=\beta F^{x,b}_\beta(\beta)+(1-2\beta)\int_{[0,\beta]} s\mathrm{d}F^{x,b}_\beta(s)=\beta+\frac{1}{2}(1-2\beta)=\frac{1}{2}.$$

This completes the proof. □

The proof of Proposition 5 consists of a series of lemmas. The next two lemmas are technical ones.

Lemma 8. *Let $f:[u,v]\to\mathbb{R}$ be a function. Suppose f is bounded from below. Then for all $x\in(u,v)$, we have*[26]

$$\mathrm{co}_{[u,v]}f(x)=\sup_{\substack{\text{linear } \ell:[u,v]\to\mathbb{R},\\ \ell\leq f}}\ell(x). \tag{30}$$

Proof. Because $\mathrm{co}_{[u,v]}f$ is convex by definition and because x is an interior point, the left derivative of $\mathrm{co}_{[u,v]}f$ at x exists. Denote this derivative by $a\in\mathbb{R}$. Then $f(x')\geq \mathrm{co}_{[u,v]}f(x')\geq \ell^x(x')$ for all $x'\in[u,v]$ where $\ell^x:[u,v]\to\mathbb{R}$ is the linear function $x'\mapsto a(x'-x)+\mathrm{co}_{[u,v]}f(x)$. Because ℓ^x is below f and because $\ell^x(x)=\mathrm{co}_{[u,v]}f(x)$ by construction, we know

$$\mathrm{co}_{[u,v]}f(x)\leq \sup_{\substack{\text{linear } \ell:[u,v]\to\mathbb{R},\\ \ell\leq f}} l(x).$$

The other direction of the above inequality is straightforward, completing the proof. □

For any real numbers $u<v$, let $C[u,v]$ be the space of all continuous functions over $[u,v]$ endowed with the uniform norm $\|\cdot\|$.

Lemma 9. *(i) Suppose $0<u<v\leq\frac{1}{2}$. Let $h\in C[u,v]$ be an arbitrary function and $T:C[u,v]\to C[u,v]$ be the operator defined as follows: for $f\in C[u,v]$,*

$$(Tf)(p)\equiv\frac{h(p)+(1-2p)\int_u^p f(s)\mathrm{d}s}{2p(1-p)},\ \forall p\in[u,v].$$

Then T is a contraction mapping. As a result, T has a unique fixed point $f^*\in C[u,v]$ and $\lim_n T^n g=f^*$ for any $g\in C[u,v]$.

[26] In general, (30) does not hold for $x\in\{u,v\}$ unless f is continuous at the endpoints.

(ii) Suppose $\frac{1}{2} \le u < v < 1$. *Let* $h \in C[u,v]$ *be an arbitrary function and* $T : C[u,v] \to C[u,v]$ *be the operator defined as follows: for* $f \in C[u,v]$,

$$(Tf)(p) \equiv \frac{h(p) - (1-2p)\int_p^v f(s)\mathrm{d}s}{2p(1-p)}, \ \forall p \in [u,v].$$

Then T *is a contraction mapping. As a result,* T *has a unique fixed point* $f^* \in C[u,v]$ *and* $\lim_n T^n g = f^*$ *for any* $g \in C[u,v]$.

Proof. We only prove *(i)*. The other case is symmetric. Clearly, for every $f \in C[u,v]$, the function Tf is well defined and continuous. Thus, T is well defined. For any $f, f' \in C[u,v]$, we have

$$\begin{aligned}\|Tf - Tf'\| &= \max_{u \le p \le v} \left| \frac{1-2p}{2p(1-p)} \int_u^p \big(f(s) - f'(s)\big)\mathrm{d}s \right| \\ &\le \|f - f'\| \max_{u \le p \le v} \frac{(1-2p)(p-u)}{2p(1-p)}.\end{aligned}$$

Because

$$\frac{(1-2p)(p-u)}{2p(1-p)} = 1 - \frac{p(1-u)+(1-p)u}{2p(1-p)} \le 1 - \frac{u(1-u)}{v(1-v)}, \ \forall p \in [u,v],$$

we know T is a contraction mapping. □

The next two lemmas show that if the conditional payoff function ϕ^F for a standard information structure $F \in \mathcal{F}$ over $[\alpha, \beta]$ is bounded from below by a linear function, then F itself over $[\alpha, \beta]$ is bounded from below by a particular function.

Lemma 10. *Suppose* $0 \le \alpha < \beta < 1$ *and* $\alpha < \frac{1}{2}$. *Let* $F \in \mathcal{F}$ *be an arbitrary standard information structure. Assume* $\ell(p) \equiv a(p-x)+b \le \phi^F(p)$ *for all* $p \in [\alpha, \beta]$ *where* $(x,b) \in A(\alpha,\beta)$ *and* $a > 0$. *If* $x > 0$, *then* $F \ge G$ *over* $[x, \min\{\frac{1}{2}, \beta\}]$ *where* $G : [x, \min\{\frac{1}{2}, \beta\}] \to \mathbb{R}$ *is defined as*

$$G(p) \equiv (1-2x)a+2b+\frac{(1-2x)b-2x(1-x)a}{2\sqrt{x(1-x)}}\frac{1-2p}{\sqrt{p(1-p)}}, \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}]. \quad (31)$$

If $x = 0$, *which only occurs when* $\alpha = 0$ *and in which case* $b = 0$, $F \ge a$ *over* $[0, \min\{\frac{1}{2}, \beta\}]$.

Proof. First consider the case where $x > 0$. Define a sequence of continuous functions $\{G_n\}_{n \ge 0}$ over $[x, \min\{\frac{1}{2}, \beta\}]$ as follows:

$$G_0(p) \equiv 0, \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}],$$

and
$$G_n(p) \equiv \frac{\ell(p) + (1-2p)\int_x^p G_{n-1}(s)\mathrm{d}s}{2p(1-p)}, \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}].$$

By Lemma 9, $\{G_n\}_{n\geq 0}$ uniformly converges to a continuous function $G : [x, \min\{\frac{1}{2}, \beta\}] \to \mathbb{R}$ that satisfies

$$G(p) = \frac{\ell(p) + (1-2p)\int_x^p G(s)\mathrm{d}s}{2p(1-p)}, \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}].$$

This functional equation defines a differential equation

$$2p(1-p)y' - (1-2p)y = \ell$$

over $[x, \min\{\frac{1}{2}, \beta\}]$ with the initial condition $y(x) = 0$. The unique solution to this differential equation is

$$G(s) \equiv (1-2x)a + 2b + \frac{(1-2x)b - 2x(1-x)a}{2\sqrt{x(1-x)}} \frac{1-2s}{\sqrt{s(1-s)}}, \ \forall s \in [x, \min\{\frac{1}{2}, \beta\}].$$

It remains to show that $F \geq G$ over $[x, \min\{\frac{1}{2}, \beta\}]$. For this, it suffices to show that $F \geq G_n$ over $[x, \min\{\frac{1}{2}, \beta\}]$ for all $n \geq 0$. We show this by induction. Clearly this is true for $n = 0$ because $F \geq 0$. Assume $F \geq G_n$ over $[x, \min\{\frac{1}{2}, \beta\}]$ for some $n \geq 0$. Observe that by integration by parts, (12) can be written as

$$\phi^F(p) = 2p(1-p)F(p) - (1-2p)\int_0^p F(s)\mathrm{d}s.$$

Then $\ell \leq \phi^F$ over $[\alpha, \beta]$ implies that for all $p \in [x, \min\{\frac{1}{2}, \beta\}] \subseteq [\alpha, \beta]$,

$$2p(1-p)F(p) \geq \ell(p) + (1-2p)\int_0^p F(s)\mathrm{d}s \geq \ell(p) + (1-2p)\int_x^p F(s)\mathrm{d}s.$$

Because $F \geq G_n$ over $[x, \min\{\frac{1}{2}, \beta\}]$ by induction hypothesis and because $(1-2p) \geq 0$ for $p \in [0, \frac{1}{2}]$, we know

$$2p(1-p)F(p) \geq \ell(p) + (1-2p)\int_x^p G_n(s)\mathrm{d}s, \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}],$$

or equivalently

$$F(p) \geq \frac{\ell(p) + (1-2p)\int_x^p G_n(s)\mathrm{d}s}{2p(1-p)} = G_{n+1}(p), \ \forall p \in [x, \min\{\frac{1}{2}, \beta\}].$$

This completes the proof for $x > 0$.

Now assume $x = 0$. Then $\ell(p) = ap$. For $n \geq 1$, define $\ell_k(p) \equiv a(p - \frac{1}{k})$. Observe that $(\frac{1}{k}, 0) \in A(0, \beta)$ when k is sufficiently large. Because $\ell_k(p) \leq \ell(p) \leq \phi^F(p)$ for all $p \in [0, \beta]$, by the previous results we know $F \geq \widetilde{G}_k$ over $[\frac{1}{k}, \min\{\frac{1}{2}, \beta\}]$ where

$$\widetilde{G}_k(p) = \left(1 - \frac{2}{k}\right) a - \sqrt{\frac{1}{k}\left(1 - \frac{1}{k}\right)} \frac{1 - 2p}{2p(1-p)}, \ \forall p \in [\frac{1}{k}, \min\{\frac{1}{2}, \beta\}].$$

Therefore, for all $p \in (0, \min\{\frac{1}{2}, \beta\}]$, we have

$$F(p) \geq \lim_{k \to \infty} \widetilde{G}_k(p) = a,$$

By right continuity of F, we know $F(0) \geq a$ as well, completing the proof. □

Lemma 11. *Suppose $0 \leq \alpha < \frac{1}{2} < \beta < 1$. Let $F \in \mathcal{F}$ be an arbitrary standard information structure. Assume $\ell(p) \equiv a(p - x) + b \leq \phi^F(p)$ for all $p \in [\alpha, \beta]$ where $(x, b) \in A(\alpha, \beta)$ and $a > 0$. Then $F \geq G$ over $[\frac{1}{2}, \beta]$ where $G : [\frac{1}{2}, \beta] \to \mathbb{R}$ is defined as*

$$G(p) \equiv (1 - 2x)a + 2b - \frac{(2(1-x)a + 2b - 1)\beta}{2\sqrt{\beta(1-\beta)}} \frac{1 - 2p}{\sqrt{p(1-p)}}, \ \forall p \in [\frac{1}{2}, \beta]. \tag{32}$$

Proof. The proof is similar to that of Lemma 10.

Define $\ell' : [\frac{1}{2}, \beta] \to \mathbb{R}$ as

$$\ell'(p) = \big(a - 2\beta + 2a(\beta - \alpha) + 2b\big)p - a\beta + \beta, \ \forall p \in [\frac{1}{2}, \beta].$$

Define a sequence of continuous functions $\{G_n\}_{n \geq 0}$ over $[\frac{1}{2}, \beta]$ as follows:

$$G_0(p) \equiv 0, \ \forall p \in [\frac{1}{2}, \beta],$$

and

$$G_n(p) \equiv \frac{\ell'(p) - (1 - 2p) \int_p^\beta G_{n-1}(s) \mathrm{d}s}{2p(1-p)}, \ \forall p \in [\frac{1}{2}, \beta].$$

By Lemma 9 again, $\{G_n\}_{n \geq 0}$ uniformly converges to a continuous function $G : [\frac{1}{2}, \beta] \to \mathbb{R}$ that satisfies

$$G(p) = \frac{\ell'(p) - (1 - 2p) \int_p^\beta G(s) \mathrm{d}s}{2p(1-p)}, \ \forall p \in [\frac{1}{2}, \beta].$$

This functional equation defines a differential equation

$$-2p(1-p)y' + (1 - 2p)y = \ell'$$

over $[\frac{1}{2}, \beta]$ with the boundary condition $y(\beta) = 0$. The unique solution to this differential equation is

$$G(p) \equiv (1-2x)a + 2b - \frac{(2(1-x)a + 2b - 1)\beta}{2\sqrt{\beta(1-\beta)}} \frac{1-2p}{\sqrt{p(1-p)}}, \ \forall p \in [\frac{1}{2}, \beta].$$

Again, it remains to show that $F \geq G$ over $[\frac{1}{2}, \beta]$. Similarly as the proof for Lemma 10, it suffices to show that $F \geq G_n$ over $[\frac{1}{2}, \beta]$ for all $n \geq 1$. This can be done again by induction. Clearly $F \geq G_0$ over $[\frac{1}{2}, \beta]$. Now assume $F \geq G_n$ over $[\frac{1}{2}, \beta]$ for some $n \geq 0$. From (12) and the fact that F is a standard information structure, we can rewrite ϕ^F as

$$\begin{aligned}
\phi^F(p) &= pF(p) + (1-2p)\left(\frac{1}{2} - \int_{(p,1]} s\mathrm{d}F(s)\right) \\
&= 2p(1-p)F(p) + (1-2p)\int_p^1 F(s)\mathrm{d}s - \frac{1}{2}(1-2p),
\end{aligned}$$

where the second equality comes from integration by parts again. The assumption that $\ell \leq \phi^F$ over $[\alpha, \beta]$ then implies that for $p \in [\frac{1}{2}, \beta] \subseteq [\alpha, \beta]$,

$$\begin{aligned}
&2p(1-p)F(p) \\
\geq &\ell(p) + \frac{1}{2}(1-2p) - (1-2p)\int_p^1 F(s)\mathrm{d}s \\
\geq &\ell(p) + \frac{1}{2}(1-2p) - (1-2p)\int_p^\beta F(s)\mathrm{d}s - (1-2p)(1-\beta)F(\beta),
\end{aligned} \tag{33}$$

where the second inequality comes from the fact that $1-2p \leq 0$ for $p \in [\frac{1}{2}, \beta]$ and the fact that $\int_p^1 F(s)\mathrm{d}s \geq \int_p^\beta F(s)\mathrm{d}s + (1-\beta)F(\beta)$. Letting $p = \beta$ in the above inequality yields

$$(1-\beta)F(\beta) \geq \ell(\beta) + \frac{1}{2}(1-2\beta).$$

Plugging this inequality back into (33), we have for $p \in [\frac{1}{2}, \beta]$,

$$\begin{aligned}
2p(1-p)F(p) &\geq \ell(p) - (1-2p)\int_p^\beta F(s)\mathrm{d}s - (1-2p)\Big(\ell(\beta) - \beta\Big) \\
&= \ell'(p) - (1-2p)\int_p^\beta F(s)\mathrm{d}s.
\end{aligned}$$

Because $F \geq G_n$ over $[\frac{1}{2}, \beta]$ by induction hypothesis, we know

$$2p(1-p)F(p) \geq \ell'(p) - (1-2p)\int_p^\beta G_n(s)\mathrm{d}s, \ \forall p \in [\frac{1}{2}, \beta],$$

or equivalently

$$F(p) \geq \frac{\ell'(p) - (1-2p)\int_p^\beta G_n(s)\mathrm{d}s}{2p(1-p)} = G_{n+1}(p),\ \forall p \in [\frac{1}{2}, \beta].$$

This completes the proof. □

With Lemmas 10 and 11, we are now ready to state the following result which is the key to Proposition 5.

Lemma 12. *Fix $0 \leq \alpha < \beta < 1$ and $\alpha < \frac{1}{2}$. For any $F \in \mathcal{F}$ and linear $\ell : [\alpha, \beta] \to \mathbb{R}$ such that $\ell \leq \phi^F|_{[\alpha,\beta]}$, there exists a pair $(x, b) \in A(\alpha, \beta)$ such that*

$$\ell(p) \leq \phi^{F_\beta^{x,b}}(p),\ \forall p \in [\alpha, \beta].$$

Proof. Fix an $F \in \mathcal{F}$ and an arbitrary linear $\ell : [\alpha, \beta] \to \mathbb{R}$ such that $\ell \leq \phi^F|_{[\alpha,\beta]}$. In what follows, we will simply write $\ell \leq \phi^F$ to mean $\ell \leq \phi^F|_{[\alpha,\beta]}$ because we only focus on functions with domain $[\alpha, \beta]$.

Recall that Kamenica and Gentzkow (2011) show that $\max_{F\in\mathcal{F}} \phi^F(\alpha) = \phi^{F_\beta^{\alpha,\alpha}}(\alpha) = \alpha$. So we know $\ell(\alpha) \leq \phi^F(\alpha) \leq \alpha$. If ℓ is nonincreasing, we then immediately know that $\ell \leq \phi^{F_\beta^{\alpha,\alpha}}$ because $\phi^{F_\beta^{\alpha,\alpha}}$ is increasing by Lemma 6 and $\ell(\alpha) \leq \alpha = \phi^{F_\beta^{\alpha,\alpha}}(\alpha)$. Similarly, if ℓ is increasing and $\ell(\beta) \leq 0$, we know $\ell \leq \phi^{F_\beta^{\alpha,\alpha}}$ again because $\ell \leq 0 \leq \phi^{F_\beta^{\alpha,\alpha}}$.

In the following, we assume that ℓ is increasing and $\ell(\beta) > 0$. If $\ell(\alpha) \geq 0$, define $x_\ell \equiv \alpha$. Observe that in this case, $\ell(x_\ell) = \ell(\alpha) \leq \phi^F(\alpha) \leq \max_{F'\in\mathcal{F}} \phi^{F'}(\alpha) = \alpha$. If $\ell(\alpha) < 0$, define x_ℓ to be the unique point in (α, β) such that $\ell(x_\ell) = 0$. In both cases, we can express ℓ as

$$\ell(p) = a(p - x_\ell) + \ell(x_\ell)$$

for some $a > 0$. We proceed by considering three different cases of α and β.

Case 1: $0 \leq \alpha < \beta \leq \frac{1}{2}$.

We proceed to show that $l \leq \phi^{F_\beta^{x_\ell,\ell(x_\ell)}}$. By Lemma 6, it suffices to show that $a \leq a_\beta^{x_\ell,\ell(x_\ell)}$. If $x_\ell = \alpha = 0$, we know $0 \leq \ell(x_\ell) \leq \phi^F(0) = 0$ where the first inequality comes from the construction of x_ℓ. Because $(0, 0) \in A(0, \beta)$, Lemma 10 implies that $F \geq a$ over $[0, \beta]$. So

$$(1-\beta)F(\beta) + \int_0^\beta F(s)\mathrm{d}s \geq (1-\beta)a + a\beta = a.$$

By (18), we have $a \leq \frac{1}{2} = a_\beta^{0,0}$.

Now assume $x_\ell > 0$. Because $(x_\ell, \ell(x_\ell)) \in A(\alpha, \beta)$, Lemma 10 again implies that $F \geq G$ over $[x_\ell, \beta]$ where

$$G(p) \equiv (1-2x_\ell)a + 2\ell(x_\ell) + \frac{(1-2x_\ell)\ell(x_\ell) - 2x_\ell(1-x_\ell)a}{2\sqrt{x_\ell(1-x_\ell)}} \frac{1-2p}{\sqrt{p(1-p)}}, \ \forall p \in [x_\ell, \beta].$$

Therefore we have

$$\begin{aligned}
&(1-\beta)F(\beta) + \int_0^\beta F(s)\mathrm{d}s \\
\geq&(1-\beta)G(\beta) + \int_{x_\ell}^\beta G(s)\mathrm{d}s \\
=&(1-x_\ell)\left[1 - \sqrt{\frac{x_\ell(1-\beta)}{(1-x_\ell)\beta}}\right]a + \frac{1}{2}\left[2 + \frac{(1-2x_\ell)\sqrt{1-\beta}}{\sqrt{x_\ell(1-x_\ell)\beta}}\right]\ell(x_\ell). \qquad (34)
\end{aligned}$$

Comparing (34) and (17) when $(x, b) = (x_\ell, \ell(x_\ell))$, it is immediately to see that if $a > a_\beta^{x_\ell, \ell(x_\ell)}$, we will have $(1-\beta)F(\beta) + \int_0^\beta F(s)\mathrm{d}s > \frac{1}{2}$ which violates (18) again. Therefore we must have $a \leq a_\beta^{x_\ell, \ell(x_\ell)}$, completing the proof of this case.

Case 2: $0 \leq \alpha < 1-\beta < \frac{1}{2} < \beta$.

Because $\phi^F(\beta) \leq \max_{F' \in \mathcal{F}} \phi^{F'}(\beta) = \frac{1}{2}$, we know $\ell(\beta) \leq \frac{1}{2}$. Because $\phi^{F_\beta^{1-\beta,0}}(\beta) = \frac{1}{2}$ by Lemma 6, if $x_\ell > 1-\beta$, we know $\ell \leq \phi^{F_\beta^{1-\beta,0}}$. Now assume $x_\ell \leq 1-\beta$. If $x_\ell = \alpha = 0$, we can use a similar proof as before to show that

$$\left(1 - \frac{1}{2}\right) F\left(\frac{1}{2}\right) + \int_0^{\frac{1}{2}} F(s)\mathrm{d}s \geq a.$$

By (18) again, we know $a \leq \frac{1}{2} = a_\beta^{0,0}$. Now assume $0 < x_\ell \leq 1-\beta$. Because $(x_\ell, \ell(x_\ell)) \in A(\alpha, \beta)$, we only need to show that $a \leq a_\beta^{x_\ell, \ell(x_\ell)}$. Lemma 10 implies that $F \geq G_1$ over $[x_\ell, \frac{1}{2}]$ where

$$G_1(p) \equiv (1-2x_\ell)a + 2\ell(x_\ell) + \frac{(1-2x_\ell)\ell(x_\ell) - 2x_\ell(1-x_\ell)a}{2\sqrt{x_\ell(1-x_\ell)}} \frac{1-2p}{\sqrt{p(1-p)}}, \ \forall p \in [x_\ell, \frac{1}{2}].$$

Moreover, Lemma 11 implies that $F \geq G_2$ over $[\frac{1}{2}, \beta]$ where

$$G_2(p) \equiv (1-2x_\ell)a + 2\ell(x_\ell) - \frac{(2(1-x_\ell)a + 2\ell(x_\ell) - 1)\beta}{2\sqrt{\beta(1-\beta)}} \frac{1-2p}{\sqrt{p(1-p)}}, \ \forall p \in [\frac{1}{2}, \beta].$$

Therefore, we have

$$
\begin{aligned}
&(1-\beta)F(\beta)+\int_0^\beta F(s)\mathrm{d}s\\
\geq&(1-\beta)G_2(\beta)+\int_{x_\ell}^{\frac{1}{2}} G_1(s)\mathrm{d}s+\int_{\frac{1}{2}}^{\beta} G_2(s)\mathrm{d}s\\
=&(1-x_\ell)\left[1-\sqrt{\frac{x_\ell(1-\beta)}{(1-x_\ell)\beta}}\right]a+\frac{1}{2}\left[2+\frac{(1-2x_\ell)\sqrt{1-\beta}}{\sqrt{x_\ell(1-x_\ell)\beta}}\right]\ell(x_\ell).
\end{aligned}
\tag{35}
$$

Similarly as previous case, comparing (35) and (17) when $(x,b)=(x_\ell,\ell(x_\ell))$, it is immediately to see that if $a>a_\beta^{x_\ell,\ell(x_\ell)}$, we will have $(1-\beta)F(\beta)+\int_0^\beta F(s)\mathrm{d}s>\frac{1}{2}$ which violates (18) again. Therefore we must have $a\leq a_\beta^{x_\ell,\ell(x_\ell)}$, completing the proof of this case.

Case 3: $1-\beta\leq\alpha<\frac{1}{2}<\beta$.

Similarly as before, because $\phi^{F_\beta^{\alpha,b_\beta^*(\alpha)}}(\beta)=\frac{1}{2}\geq\phi^F(\beta)\geq\ell(\beta)$, if either $x_\ell=\alpha$ and $\ell(x_\ell)<b_\beta^*(\alpha)$, or $x_\ell>\alpha$ and $\ell(x_\ell)=0$, we know $\ell\leq\phi^{F_\beta^{\alpha,b_\beta^*(\alpha)}}$. Now assume $x_\ell=\alpha$ and $\ell(x_\ell)\geq b_\beta^*(\alpha)$. Because again $\ell(x_\ell)\leq\phi^F(\alpha)\leq\alpha$, we know $(x_\ell,\ell(x_\ell))\in A(\alpha,\beta)$. Then we can use a similar argument as in the previous case to show that $a\leq a_\beta^{x_\ell,\ell(x_\ell)}$, completing the proof of this case. □

We are now ready to prove Proposition 5.

Proof for Proposition 5. For any $F\in\mathcal{F}$, Lemmas 8 and 12 together imply

$$
\mathrm{co}_{[\alpha,\beta]}\phi^F(\pi)=\sup_{\substack{\text{linear }\ell:[\alpha,\beta]\to\mathbb{R}\\ \ell\leq\phi^F}}\ell(\pi)\leq\max_{(x,b)\in A(\alpha,\beta)}\phi^{F_\beta^{x,b}}(\pi)=V_{\alpha,\beta}^*(\pi),\ \forall\pi\in(\alpha,\beta).
$$

Hence $V([\alpha,\beta],\pi)=\max_{F\in\mathcal{F}}\mathrm{co}_{[\alpha,\beta]}\phi^F(\pi)=V_{\alpha,\beta}^*(\pi)$ for all $\pi\in(\alpha,\beta)$. □

Proof for Proposition 6. Because of Proposition 5, to show an information structure $F\in\{F_\beta^{x,b}\}_{(x,b)\in A(\alpha,\beta)}$ is the sender's optimal information structure, it suffices to show that this F is optimal among $\{F_\beta^{x,b}\}_{(x,b)\in A(\alpha,\beta)}$.

First assume $0<\alpha<\beta\leq\frac{1}{2}$ or $0<\alpha<1-\beta<\frac{1}{2}<\beta$. Because $a_\beta^{\alpha,b}$ defined in (17) is linear in $b\in[0,\alpha]$, it is straightforward to see that the linear functions $\{\phi^{F_\beta^{\alpha,b}}\}_{b\in[0,\alpha]}$ all intersect at $\pi_{\alpha,\beta}^*\in(\alpha,\beta)$ defined in (24). It is then obvious to see that for all $b\in[0,\alpha)$, $\phi^{F^{\alpha,\alpha}}(p)\geq\phi^{F_\beta^{\alpha,b}}(p)$ for all $p\in[\alpha,\pi_{\alpha,\beta}^*]$ with strict inequality if $p<\pi_{\alpha,\beta}^*$. Moreover, it is also straightforward to verify that $\phi^{F_\beta^{\alpha,0}}(\pi_{\alpha,\beta}^*)=a_\beta^{\alpha,0}(\pi_{\alpha,\beta}^*-\alpha)>a_\beta^{x,0}(\pi_{\alpha,\beta}^*-x)=\phi^{F_\beta^{x,0}}(\pi_{\alpha,\beta}^*)$ for all $x\in(\alpha,\min\{\beta,1-\beta\})$. Thus, for all $x \in (\alpha, \min\{\beta, 1-\beta\})$, we have $\phi^{F^{\alpha,\alpha}}(p) > \phi^{F_\beta^{x,0}}(p)$ for all $p \in [\alpha, \pi^*_{\alpha,\beta}]$. Therefore, we know $F^{\alpha,\alpha}$ is optimal for all $\pi \in (\alpha, \pi^*_{\alpha,\beta}]$. On the other hand, the fact that $\{\phi^{F_\beta^{\alpha,b}}\}_{b\in[0,\alpha]}$ all intersect at $\pi^*_{\alpha,\beta}$ also implies that for all $b \in (0, \alpha]$, $\phi^{F_\beta^{\alpha,0}}(p) > \phi^{F_\beta^{\alpha,b}}(p)$ for all $p \in (\pi^*_{\alpha,\beta}, \beta]$. Thus, $\{F_\beta^{\alpha,b}\}_{b\in(0,\alpha]}$ are not optimal when $\pi > \pi^*_{\alpha,\beta}$. This implies that

$$V^*_{\alpha,\beta}(\pi) = \max_{x\in[\alpha,\min\{\beta,1-\beta\})} \phi^{F_\beta^{x,0}}(\pi) = \max_{x\in[\alpha,\min\{\beta,1-\beta\})} a_\beta^{x,0}(\pi - x),\ \forall \pi \in (\pi^*_{\alpha,\beta}, \beta).$$

Plugging in the expression for $a_\beta^{x,0}$ yields

$$V^*_{\alpha,\beta}(\pi) = \max_{x\in[\alpha,\min\{\beta,1-\beta\})} \frac{\pi - x}{2(1-x)\left[1 - \sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right]},\ \forall \pi \in (\pi^*_{\alpha,\beta}, \beta). \tag{36}$$

It is then easy to see that for each $\pi \in (\pi^*_{\alpha,\beta}, \beta)$, there exists a unique solution $x^*(\pi) \in (\alpha, \min\{\beta, 1-\beta\})$ and thus $F_\beta^{x^*(\pi),0}$ is optimal. Furthermore, simple algebra shows that $x^* : (\pi^*_{\alpha,\beta}, \beta) \to (\alpha, \min\{\beta, 1-\beta\})$ is strictly increasing and onto. The above analysis assumed $\alpha > 0$. When $\alpha = 0$, $\pi^*_{\alpha,\beta} = 0$ by construction and we only need to consider (36).

Finally, assume $1 - \beta \le \alpha < \frac{1}{2} < \beta$. Similarly as above, we know $\{\phi^{F_\beta^{\alpha,b}}\}_{b\in[b^*_\beta(\alpha),\alpha]}$ intersect at $\pi^*_{\alpha,\beta} \in (\alpha, \beta)$. Thus $\phi^{F^{\alpha,\alpha}}(\pi)$ is the biggest for $\pi < \pi^*_{\alpha,\beta}$ and $\phi^{F_\beta^{\alpha,b^*_\beta(\alpha)}}(\pi)$ is the biggest if $\pi > \pi^*_{\alpha,\beta}$. Therefore, $F^{\alpha,\alpha}$ is optimal if $\pi \in (\alpha, \pi^*_{\alpha,\beta})$ and $F_\beta^{\alpha,b^*_\beta(\alpha)}$ is optimal if $\pi \in (\pi^*_{\alpha,\beta}, \beta)$. □

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